

Wild monodromy action on the character variety of the fifth Painlevé equation

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Abstract

The (wild) character variety of the Painlevé V equation is constructed from the character varieties of Painlevé VI through confluence. The known description of the action of the monodromy of Painlevé VI on the corresponding character variety is used to construct a 1-parameter group action on the character variety of Painlevé V. It is expected that this action corresponds to the wild monodromy of Painlevé V, that is the group generated by the monodromy, the exponential torus and the nonlinear Stokes operators of the corresponding Hamiltonian foliation.

Key words: Painlevé equations, character varieties, Stokes phenomenon, wild monodromy, confluence

1 Introduction

One approach to the Painlevé equations is through the study of isomonodromic deformations of meromorphic connections on \mathbb{CP}^1 . It is well known that the sixth Painlevé equation P_{VI} governs the isomonodromic deformations of Garnier systems (2×2 traceless linear differential systems) with 4 Fuchsian singular points $0, 1, t, \infty$ (t being the independent variable). It also governs isomonodromic deformations of 3×3 systems in Okubo normal form, and their dual through the Laplace transform, 3×3 systems in Birkhoff normal form with an irregular singularity of Poincaré rank 1 at the origin (of eigenvalues $0, 1, t$) and a Fuchsian singularity at the infinity. The fifth Painlevé equation is obtained from P_{VI} through the change of the independent variable $t \mapsto 1 + \epsilon t$, accompanied by a suitable change of parameters, and the confluence $\epsilon \rightarrow 0$. It is equivalent to the isomonodromy problems for the corresponding limit confluent linear systems. In this setting, the Riemann-Hilbert correspondence can be interpreted as a map between space of local solutions of the given Painlevé equation with fixed values of parameters, and the space of generalized monodromy representations (monodromy & Stokes data) with fixed local multipliers. This latter space is called the character variety. Hence, the Riemann-Hilbert correspondence conjugates the transcendental flow of the Painlevé equations to a locally constant flow on the moduli spaces of monodromy representations (character varieties) [IIS06]. It is well-known that one can use this correspondence in order to describe the non-linear monodromy action of the Painlevé equation, i.e. the action of analytic continuation of solutions along loops in the independent variable (time variable). In the case of P_{VI} this description of the non-linear monodromy action on the equation is classical: it corresponds to a braid group action on the character variety [Iwa03, Iwa02]. The study of this action has many important applications, such as a construction and classification of algebraic solutions of P_{VI} [Boa10, LT08], or a proof of its irreducibility in the sense of Malgrange [CL09]. For the equation P_V the monodromy alone does not carry enough information about the equation, one must also take in account the nonlinear Stokes phenomenon of the underlying Hamiltonian foliation at its irregular singularity at the infinity and introduce the notion of a wild monodromy.

The goal of this article is to connect the character variety of P_V with that of P_{VI} through a confluence, and to describe the wild monodromy action on it. Both the study of the confluence of character varieties and the description of the wild monodromy of P_V seems to be completely new.

This article fits into the general program of study of “wild monodromy” actions on the character varieties of isomonodromic deformations of linear differential systems that was sketched in [PR15].

1.1 The Painlevé equations.

The Painlevé equations originated from the effort of classifying all second order ordinary differential equations of type $q'' = R(q', q, t)$, with R rational, possessing the so called *Painlevé property* which controls the ramification points of solutions:

Painlevé property: *Each germ of a solution can be meromorphically continued along any path avoiding the singular points of the equation. In other words, solutions cannot have any other movable singularities other than poles.*

Painlevé and Gambier [Gam10] produced a list of 50 canonical forms of equations to which any such equation can be reduced. Aside of equations corresponding to various classical special functions, the list contained six new families of equations, P_I, \dots, P_{VI} , whose general solutions provided a new kind of special functions. In many aspects they may be regarded as non-linear analogues of the hypergeometric equations [IKSY91]. The equation P_{VI} is a mother equation for the other Painlevé equations, which can be obtained through degenerations and confluenes following the diagram [OO06]

$$\begin{array}{ccccccc}
 & & & P_{III}^{D_6} & \rightarrow & P_{III}^{D_7} & \rightarrow & P_{III}^{D_8} \\
 & & \nearrow & & \searrow & & & \\
 P_{VI} & \rightarrow & P_V & \rightarrow & P_V^{deg} & \rightarrow & P_I & \\
 & & \searrow & & \nearrow & & & \\
 & & & P_{IV} & \rightarrow & P_{II}^{FN} & &
 \end{array}$$

This article studies some aspects of the confluence $P_{VI} \rightarrow P_V$ through the Riemann-Hilbert correspondence.

Each of the Painlevé equations is equivalent to a time dependent *Hamiltonian system*

$$\frac{dq}{dt} = \frac{\partial H_{\bullet}(q, p, t)}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H_{\bullet}(q, p, t)}{\partial q}, \quad \bullet = I, \dots, VI, \quad (1)$$

with a polynomial Hamiltonian function, from which it is obtained by reduction to the variable q .

The general form of the sixth Painlevé equation is

$$\begin{aligned}
 P_{VI}: \quad q'' &= \frac{1}{2} \left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t} \right) (q')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t} \right) q' \\
 &+ \frac{q(q-1)(q-t)}{2t^2(t-1)^2} \left[(\vartheta_{\infty}-1)^2 - \vartheta_0^2 \frac{t}{q^2} + \vartheta_1^2 \frac{(t-1)}{(q-1)^2} + (1-\vartheta_t^2) \frac{t(t-1)}{(q-t)^2} \right],
 \end{aligned}$$

where $\vartheta = (\vartheta_0, \vartheta_t, \vartheta_1, \vartheta_{\infty}) \in \mathbb{C}^4$ are complex constants. Its Hamiltonian function is given by

$$\begin{aligned}
 H_{VI} &= \frac{1}{t(t-1)} \left[q(q-1)(q-t)p^2 - \left(\vartheta_0(q-1)(q-t) + \vartheta_1 q(q-t) + (\vartheta_t-1)q(q-1) \right) p \right. \\
 &\quad \left. + \frac{1}{4} \left((\vartheta_0 + \vartheta_1 + \vartheta_t - 1)^2 - \vartheta_{\infty}^2 \right) (q-t) \right].
 \end{aligned}$$

The Hamiltonian system of P_{VI} has three simple (regular) singular points on the Riemann sphere \mathbb{CP}^1 at $t = 0, 1, \infty$.

The fifth Painlevé equation P_V is obtained from P_{VI} by the change of the independent variable

$$t \mapsto 1 + \epsilon t, \quad \vartheta_t \mapsto \frac{1}{\epsilon}, \quad \vartheta_1 \mapsto -\frac{1}{\epsilon} + \tilde{\vartheta}_1 + 1,$$

which sends the three singularities to $-\frac{1}{\epsilon}, 0, \infty$, and then by taking the limit $\epsilon \rightarrow 0$, so that the two simple singular points $-\frac{1}{\epsilon}$ and ∞ merge into a double (irregular) singularity at the infinity:¹

$$P_V : \quad q'' = \left(\frac{1}{2q} + \frac{1}{q-1} \right) (q')^2 - \frac{1}{t} q' + \frac{(q-1)^2}{2t^2} \left((\vartheta_\infty - 1)^2 q - \frac{\vartheta_0^2}{q} \right) + (1 + \tilde{\vartheta}_1) \frac{q}{t} - \frac{q(q+1)}{2(q-1)}.$$

The same confluence procedure applied to H_{VI} produces the corresponding polynomial Hamiltonian:

$$H_V = \frac{1}{t} \left[q(q-1)^2 p^2 - \left(\vartheta_0(q-1)^2 + \tilde{\vartheta}_1 q(q-1) + tq \right) p + \frac{1}{4} \left((\vartheta_0 + \tilde{\vartheta}_1)^2 - \vartheta_\infty^2 \right) (q-1) \right].$$

1.2 Monodromy of P_{VI} .

Let $\mathcal{M}_{VI}(\vartheta)$ be the (q, p, t) -space of the foliation given the Hamiltonian system of P_{VI} with a parameter $\vartheta = (\vartheta_0, \vartheta_t, \vartheta_1, \vartheta_\infty)$; and let $\mathcal{M}_{VI,t}(\vartheta)$ be its fibre with respect to the projection $(q, p, t) \mapsto t$, which is transversal to the foliation. Naively, $\mathcal{M}_{VI,t}(\vartheta)$ would be the (q, p) -space \mathbb{C}^2 , but one must take in account that the solutions have poles, therefore to define the foliation, it is necessary to add a bunch of points “at infinity”. Hence, the good space

$$\mathcal{M}_{VI,t}(\vartheta) = \text{the Okamoto space of initial conditions},$$

is rather some complex surface (an 8-point blow-up of the Hirzebruch surface Σ_2 minus an anti-canonical divisor, see [Oka79] and [IIS06], section 3.6).

The Painlevé property of P_{VI} means that there is a well-defined non-linear *monodromy* action on the foliation

$$\pi_1(\mathbb{CP}^1 \setminus \{0, 1, \infty\}, t_0) \rightarrow \text{Aut}(\mathcal{M}_{VI,t_0}(\vartheta)),$$

where the flow of the vector field (1) along each loop in the t -space $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ with a base-point t_0 gives rise to an automorphism of $\mathcal{M}_{VI,t}(\vartheta)$. (See Figure 1.)

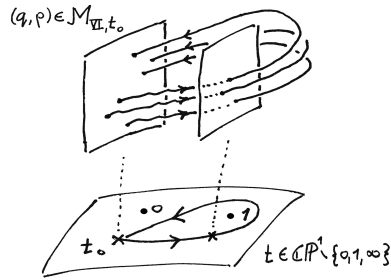


Figure 1: The Painlevé flow and its monodromy.

In case of P_{VI} this monodromy carries a lot of important information about the equation. For example, algebraic solutions of P_{VI} , which are very important in physical applications, correspond to finite orbits of the monodromy action (see e.g. [Boa10, LT08])). The monodromy is also very important from the point of view of the differential Galois

¹The equation (1.1) is the fifth equation of Painlevé with a parameter $\eta_1 = -1$. A general form of this equation is obtained by a further change of variable $t \mapsto -\eta_1 t$. The degenerate case P_V^{deg} with $\eta_1 = 0$ which has only a regular singular point at ∞ is not considered here.

theory: the Galois D-groupoid of Malgrange of the foliation is “generated” by the monodromy. Cantat and Loray [CL09] have showed the irreducibility of P_{VI} in the sense of Malgrange, i.e. maximality of the Malgrange groupoid (which implies transcendentness of general solutions [Cas09]), by studying the action of this monodromy.

Similarly, there is a well-defined monodromy action also on P_V and other Painlevé equations. However the confluence of the two singularities $-\frac{1}{\epsilon}, \infty$ in the confluent family of P_{VI} into a single singularity ∞ of P_V means that an important part of the monodromy group is lost and therefore also the information carried by it. This lost information reappears in the (non-linear) Stokes phenomenon at the irregular singularity.

1.3 Nonlinear Stokes phenomenon of P_V .

In the local coordinate $s = t^{-1}$ at $t = \infty$, the Hamiltonian system of P_V has the form

$$s^2 \frac{d}{ds} \begin{pmatrix} q \\ p \end{pmatrix} = s \begin{pmatrix} \vartheta_0 \\ \frac{(\vartheta_0 + \tilde{\vartheta}_1)^2 - \vartheta_\infty^2}{4} \end{pmatrix} + \begin{pmatrix} 1 + \omega s \\ -1 - \omega s \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} - s \begin{pmatrix} -(\vartheta_0 + \tilde{\vartheta}_1)q^2 + 2pq - 4p^2 + 2pq^3 \\ 2(\vartheta_0 + \tilde{\vartheta}_1)pq - p^2 + 4p^2q - 3p^2q^2 \end{pmatrix}, \quad (2)$$

$\omega = -2\vartheta_0 - \tilde{\vartheta}_1$ (cf. [Tak83]), with a resonant irregular singularity at $s = 0$.

Proposition 1 (Takano [Tak83], Yoshida [Yos85]). *The above system can be brought to a formal normal form*

$$s^2 \frac{d}{ds} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (1 + \omega s + 4v_1v_2s) \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}, \quad (3)$$

through an essentially unique sectoral change of coordinates

$$(q, sp) = T_\pm(s, v_1, sv_2, v_1v_2),$$

bounded and analytic on a cone in the v -variable above two sectors in the s -domain

$$s \in \Sigma_\pm = \{|\arg s \mp \frac{\pi}{2}| < \pi - \epsilon_0, |s| < \rho_0\}, \quad |v_1| < \rho_1, |v_2s| < \rho_2, |v_1v_2| < \rho_3,$$

for any $0 < \epsilon < \frac{\pi}{2}$ and some $\rho_l > 0$, $l = 0, \dots, 3$, which has an analytic expansion $T_\pm(s, v_1, sv_2, v_1v_2) = \sum_{i,j,k} T_\pm^{(i,j,k)}(s) v_1^i (sv_2)^j (v_1v_2)^k$ with respect to (v_1, v_2s, v_1v_2) whose coefficients $T_\pm^{(i,j,k)}(s)$ have asymptotic expansions in s .

The system (3), which is Hamiltonian for the Hamiltonian function

$$H(v_1, v_2, s) = \frac{(1 + \omega s)v_1v_2 + 2s(v_1v_2)^2}{s^2}$$

with respect to the standard symplectic form $dv_1 \wedge dv_2$, has a 2-parameter family of solutions

$$\begin{aligned} v_1 &= C_1 e^{-\frac{1}{s}} s^{\omega+4C_1C_2}, \\ v_2 &= C_2 e^{\frac{1}{s}} s^{-\omega-4C_1C_2}, \end{aligned}$$

$C_1, C_2 \in \mathbb{C}$, and a first integral $C_1C_2 = v_1v_2$. The constants

$$\begin{aligned} C_1 &= v_1 e^{\frac{1}{s}} s^{-\omega-4v_1v_2}, \\ C_2 &= v_2 e^{-\frac{1}{s}} s^{\omega+4v_1v_2}, \end{aligned}$$

are local first integrals of the system, defining coordinates on the space of leaves of the foliation (3) above each of the sectors Σ_\pm .

The Hamiltonian system (3) is preserved by the one-parameter group of symplectomorphisms

$$E(\kappa) : (v_1, v_2) \mapsto (v_1\kappa, \frac{v_2}{\kappa}), \quad \kappa \in \mathbb{C}^*,$$

which are given by the flow of the Hamiltonian vector field

$$v_1\partial_{v_1} - v_2\partial_{v_2}$$

commuting with (3). We call the action of the family of automorphisms $E(\kappa)$ “an exponential torus action”.

Remark 2. A recent works of Bittmann on a formal and sectoral normalization [Bit15, Bit16] of this kind of systems provides an improved result compared to [Yos85]. It investigates transformations by a similar change of variables, defined on a polydisc in (v_1, v_2) above the sectors Σ_{\pm} , providing also a finer formal classification in which the term v_1v_2 in (3) is replaced by a function $c(v_1v_2)$. It also shows that the corresponding coefficients $T_{\pm}^{(i,j,k)}(s)$ in the analytic expansion of the transformation are not only asymptotic to those of the formal transformation but they are in fact their Borel sums.

The pullback of the coordinates (C_1, C_2) on the foliation (3) through the sectoral transformation T_{\pm} then provides local coordinates on the restriction of the original foliation (2) to Σ_{\pm} . On each of the two intersections of the overlapping sectors Σ_+, Σ_- , the map $T_{\pm} \circ T_{\mp}^{-1}$ that associates to a leaf of (2) of coordinate (C_1, C_2) on Σ_+ the leaf of the same coordinates on Σ_- (or vice-versa) provides an automorphism of the foliation. These are the non-linear *Stokes operators*. Changing on one of the sectors Σ_{\pm} the coordinate by the action of $E(\kappa)$, which happens for example if one changes the determination of $\log s$ on the sector, will produce a different pair of Stokes operators. This means that there is a parametric family of (pairs of) Stokes operators

$$T_{\pm} \circ E(\kappa) \circ T_{\mp}^{-1}$$

depending on the parameter $\kappa \in \mathbb{C}^*$. Let us remark, that unlike the monodromy, these Stokes operators are defined only locally on the foliation.

The (pseudo)-group action on the foliation of P_V generated by both the usual monodromy and the Stokes operators is the *wild monodromy* of the title of this paper.

The central question of our work is how can one relate this wild monodromy of P_V with the monodromy of P_{VI} ?

It is not very surprising that the only monodromy operators that converge are those corresponding to the loops in $\pi_1(\mathbb{CP}^1 \setminus \{-\frac{1}{\epsilon}, 0, \infty\}, t_0)$ that persist when $\epsilon \rightarrow 0$ as loops in $\pi_1(\mathbb{CP}^1 \setminus \{0, \infty\}, t_0)$, while those corresponding to the vanishing loops diverge. Indeed, for each $\epsilon \neq 0$ the monodromy group is discretely generated, while for $\epsilon = 0$ the wild monodromy group is generated by a continuous family. However, we believe that the wild monodromy is obtained by the accumulation of the monodromies when $\epsilon \rightarrow 0$.

Conjecture 3. *The monodromy of P_{VI} in the confluent family possesses limits along the sequences $\{\epsilon_n\}_{n \in \pm\mathbb{N}}$,*

$$\frac{1}{\epsilon_n} = \frac{1}{\epsilon_0} + n. \tag{4}$$

and the monodromy accumulates toward the family of such limits, depending on the parameter ϵ_0 . This family generates the wild monodromy group of P_V .

The proof of this conjecture will be the subject of a subsequent work. This article shows how one can use it to describe the wild monodromy of P_V .

1.4 Contents of the article.

The content of this article is the following: In Section 2, we describe confluence of singularities of linear systems: two simple (Fuchsian) singularities merging into a double (irregular) singularity, which provides a motivation for and a guide to understanding of the above Conjecture.

In Section 3, we will study the confluence $P_{VI} \rightarrow P_V$ through the Riemann–Hilbert correspondence, using the usual approach of isomonodromic deformations of 2×2 linear systems with four Fuchsian singularities, and the description of confluence of Section 2. We will show that the character variety (i.e. the space of equivalence classes of monodromy representations) corresponding to the equation Painlevé V is obtained from the character varieties of P_{VI} by birational changes of coordinates. Under the premise of the Conjecture we provide an explicit description of the action of the wild monodromy on the constructed character variety of P_V . These actions correspond to limits of some of the modular actions on the character variety of P_{VI} along different sequences $\epsilon_n \rightarrow 0$ (4).

In the Appendix, we show how one can obtain the same description of the character varieties but this time using isomonodromic deformations 3×3 linear systems in Birkhoff normal form with an irregular singularity of Poincaré rank 1 at the origin and a Fuchsian singularity at infinity. These systems can be obtained from the 2×2 linear systems by a middle convolution and a Laplace transform [Bo14, HF07]. The degeneration from P_{VI} to P_V corresponds to a confluence of eigenvalues. This description follows up on the previous study by Boalch [Boa05].

2 Confluence of singularities in linear systems

Consider a confluence of two regular singular points to a non-resonant irregular singular point in a family of linear systems

$$\frac{dy}{dx} = \frac{A(x, \epsilon)}{x(x - \epsilon)} y, \quad y \in \mathbb{C}^2, \quad (5)$$

with A a 2×2 traceless complex matrix depending analytically on $(x, \epsilon) \in (\mathbb{C} \times \mathbb{C}, 0)$, such that $A(0, 0) \neq 0$ has two distinct eigenvalues $\pm \lambda^{(0)}(0)$. For each fixed parameter ϵ the local differential Galois group (the Picard-Vessiot group) of the system is the Zariski closure of the group generated by

- $\epsilon \neq 0$: the monodromy around the two singular points 0 and ϵ ,
- $\epsilon = 0$: by the Stokes operators, the formal monodromy, and the exponential torus [MR91, SP03].

The question is how are these two different descriptions related?

The above kind of confluence (5) have been studied by many authors, including Garnier [Gar19], Ramis [Ram89], Schäfer [Sch98], Duval [Du98], Glutsyuk [Glu99], Zhang [Zha96], etc., here we follow mainly the approach of Parise [Par01] and of Lambert and Rousseau [LR12] (see also [HLR13, Kli16]).

Proposition 4 (Parise, Lambert, Rousseau). ²

The family (5) is formally equivalent, by means of a formal transformation $y = \hat{T}(x, \epsilon)\phi$, $\hat{T} \in \text{GL}_2(\mathbb{C}[[x, \epsilon]])$, to a model family of diagonal systems

$$\frac{d\phi}{dx} = \frac{\Lambda(x, \epsilon)}{x(x - \epsilon)} \phi, \quad \Lambda(x, \epsilon) = \Lambda^{(0)}(\epsilon) + x\Lambda^{(1)}(\epsilon) = \begin{pmatrix} \lambda(x, \epsilon) & 0 \\ 0 & -\lambda(x, \epsilon) \end{pmatrix}, \quad (6)$$

²See [HLR13] for a more general version of the theorem.

where $\pm\lambda(x, \epsilon) = \pm(\lambda^{(0)}(\epsilon) + x\lambda^{(1)}(\epsilon))$ are the eigenvalues of $A(x, \epsilon)$ modulo $x(x - \epsilon)$.
There also exist bounded analytic normalizing transformations between the systems

$$y = T^\pm(x, \epsilon)\phi$$

defined on a family of domains (Figure 2), where ϵ is from one of two sectors of opening almost 2π bisected by $\pm\lambda^{(0)}(0)\mathbb{R}^+$ respectively, and x from the domain spanned by the real trajectories of the vector fields

$$\pm e^{i\omega} \frac{x(x - \epsilon)}{2\lambda(x, \epsilon)} \partial_x$$

within some fixed neighborhood of the origin, where $\omega \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ is allowed to vary continuously as long as the phase portrait of the vector field does not bifurcate. The limit domain for $\epsilon = 0$ consists of two overlapping sectors of opening $> \pi$, and the restriction of $T^\pm(x, 0)$ to each of them is the Borel sum of the formal transformation $\hat{T}(x, 0)$.

The diagonal model family (6) has a canonical fundamental matrix solution

$$\Phi(x, \epsilon) = \begin{cases} \left(\frac{x-\epsilon}{x}\right)^{\frac{\Lambda^{(0)}(\epsilon)}{\epsilon}} (x - \epsilon)^{\Lambda^{(1)}(\epsilon)}, & \epsilon \neq 0 \\ e^{-\frac{\Lambda^{(0)}(0)}{x}} x^{\Lambda^{(1)}(0)}, & \epsilon = 0, \end{cases} \quad (7)$$

whose monodromy matrices around the points 0, ϵ , and both 0, ϵ are respectively N_0 , $N_0^{-1}N$ and N :

$$N_0(\epsilon) = e^{-2\pi i \frac{\Lambda^{(0)}(\epsilon)}{\epsilon}}, \quad N(\epsilon) = e^{2\pi i \Lambda^{(1)}(\epsilon)}.$$

Correspondingly, the family (5) has a canonical fundamental matrix solutions

$$Y^\pm(x, \epsilon) = T^\pm(x, \epsilon)\Phi(x, \epsilon),$$

In order for $Y^\pm(x, \epsilon)$ to have a limit when $\epsilon \rightarrow 0$, one has to split the domain of T^\pm in two parts by a cut between the points 0, ϵ , and take Φ_α , $\Phi_{\alpha+\pi}$ two branches of Φ (7), such that $\Phi_{\alpha+\pi}$ is a counter-clockwise continuation of Φ_α along a path exterior with respect to the two singularities (the α here is a direction of summability of $\hat{T}(x, 0)$).

There are three intersections of between the two parts of each domain and correspondingly three connection matrices between $Y_\alpha^\pm = T^\pm\Phi_\alpha$ and $Y_{\alpha+\pi}^\pm = T^\pm\Phi_{\alpha+\pi}$, see Figure 2:

- $S_R^+(\epsilon)$ (resp. $S_R^-(\epsilon)$) on the intersection sector attached to $x_R^+ = \epsilon$ (resp. $x_R^- = 0$): Φ is continued analytically there, therefore it corresponds to the ramification of T^\pm . $S_R^+(0) = S_R^-(0)$ is a Stokes matrix of the limit system.
- $N(\epsilon)S_L^+(\epsilon)$ (resp. $N(\epsilon)S_L^-(\epsilon)$) on the intersection sector attached to $x_L^+ = 0$ (resp. $x_L^- = \epsilon$): $N(\epsilon)$ is the monodromy matrix of Φ around both x_L^\pm, x_R^\pm , $S_L^+(0) = S_L^-(0)$ is a Stokes matrix of the limit system.
- $N_0(\epsilon)^{-1}N(\epsilon)$ (resp. N_0) on the center part for $\epsilon \neq 0$: the transformation T^\pm has no ramification there, therefore this connection matrix comes purely from Φ as its monodromy around the point x_R^\pm .

The matrices $S_L^\pm(\epsilon)$, $S_R^\pm(\epsilon)$, called the “*unfolded Stokes matrices*”, converge when $\epsilon \rightarrow 0$, and so does the formal monodromy $N(\epsilon)$. The formal monodromy matrix $N_0(\epsilon)$ is the source of divergence.

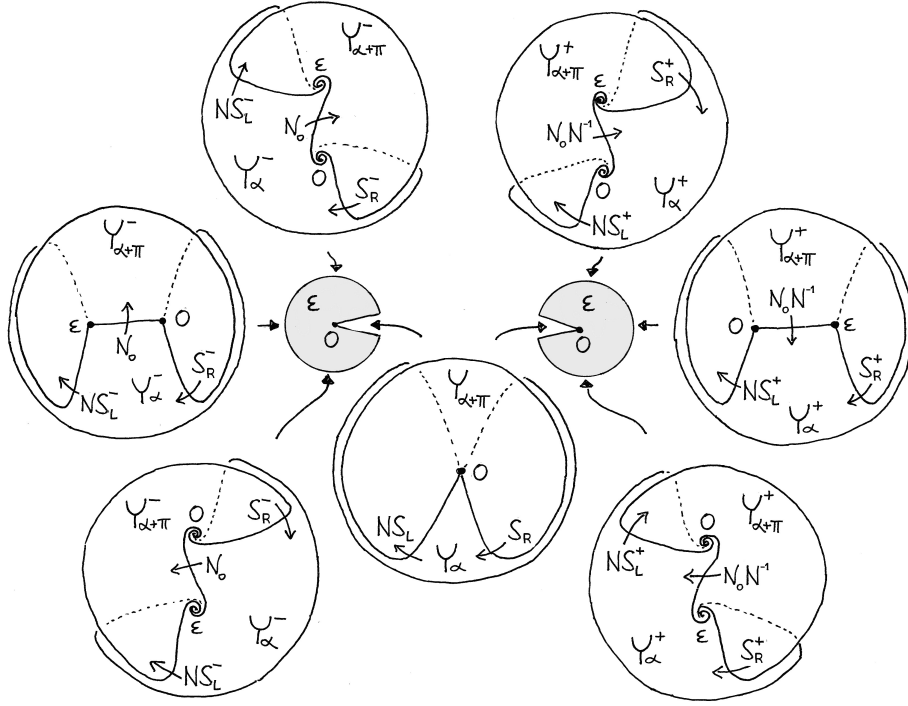


Figure 2: The fundamental matrix solutions of (5) and the unfolded Stokes matrices between them for $\lambda^{(0)}(0) \in \mathbb{R}^+$.

The Differential Galois group: Let \mathbb{K} be the field of meromorphic functions of the variable x on a fixed small neighbourhood of 0, equipped with the differentiation $\frac{d}{dx}$. For a fixed small ϵ , the *differential Galois group* of the system (5) is the group of automorphisms of the differential field $\mathbb{K}\langle Y_{\cdot}^{\pm}(\cdot, \epsilon) \rangle$, generated by the components of the fundamental matrix solution Y_{\cdot}^{\pm} , over \mathbb{K} . Since its action must preserve the system (5), it acts on it by automorphisms. Namely it acts by left multiplication on the solution space (i.e. on the foliation associated to the system). Fixing a fundamental matrix solution, say Y_{\cdot}^{\pm} , then each such automorphism is represented by a right multiplication of Y_{\cdot}^{\pm} by a constant invertible matrix, hence the differential Galois group is represented by an (algebraic) subgroup of $GL_2(\mathbb{C})$ acting on the right.

- For $\epsilon \neq 0$ from the corresponding sector the monodromy matrices of $Y_{\alpha+\pi}^+$ around the points $x_L^+ = 0$ and $x_R^+ = \epsilon$ in positive direction are

$$M_L^+(\epsilon) = N_0(\epsilon)S_L^+(\epsilon), \quad M_R^+(\epsilon) = S_R^+(\epsilon)N(\epsilon)N_0(\epsilon)^{-1}. \quad (8)$$

The representation of the differential Galois group of (5) with respect to $Y_{\alpha+\pi}^+$ is a Zariski closure of the group generated by these two monodromies. The matrix $N_0(\epsilon)$ diverges when $\epsilon \rightarrow 0$, therefore so do the monodromies $M_L^+(\epsilon)$ and $M_R^+(\epsilon)$.

- For $\epsilon = 0$, the representation of the differential Galois group of (5) with respect to $Y_{\alpha+\pi}^+$ is a Zariski closure of the wild monodromy group (cf. [MR91]), i.e. of the group generated by the Stokes matrices $S_L(0), S_R(0)$, and the exponential torus (which in this case contains also the formal monodromy):

$$E(\kappa) = \begin{pmatrix} \kappa & \\ & \kappa^{-1} \end{pmatrix}, \quad \kappa \in \mathbb{C}^*,$$

whose action $\phi \mapsto \phi E(\kappa)$ is infinitesimally generated by the vector field $\phi_1 \partial_{\phi_1} - \phi_2 \partial_{\phi_2}$. For any element $E(\kappa)$ of the exponential torus the pair of the connection matrices

between $Y_{\alpha+\pi}^+$ and $Y_{\alpha}^+E(\kappa)$ on the two intersection sectors are

$$E(\kappa)^{-1}N(0)S_L(0), \quad S_R(0)E(\kappa)$$

which belong to the representation of the Galois group. We call them *wild monodromy matrices*. The infinite family of such pairs of wild monodromy matrices depending on $\kappa \in \mathbb{C}^*$ generates the wild monodromy group.

While the monodromies $M_L^+(\epsilon)$, $M_R^+(\epsilon)$ diverge, there are particular sequences of values of ϵ tending to 0, along which the formal monodromy $N_0(\epsilon)$ stays constant and therefore such particular limits exist. Supposing, for example, that the eigenvalues $\pm\lambda = \pm(\lambda^{(0)} + x\lambda^{(1)})$ are independent of ϵ , let $\epsilon_0 \in \mathbb{C} \setminus \{0\}$, and define ϵ_n by $\frac{1}{\epsilon_n} = \frac{1}{\epsilon_0} + \frac{n}{\lambda^{(0)}}$, $n \in \mathbb{N}$, then $N_0(\epsilon_n) = N_0(\epsilon_0)$, which belongs to the exponential torus, and therefore

$$\lim_{n \rightarrow +\infty} M_L^+(\epsilon_n) = N_0(\epsilon_0)S_L(0), \quad \lim_{n \rightarrow +\infty} M_R^+(\epsilon_n) = S_R(0)N(0)N_0(\epsilon_0)^{-1}$$

is a pair of wild monodromy matrices. The family of such limits depending on ϵ_0 generates the representation of the Galois group of the limit system.

Proposition 5. *The wild monodromy group of the limit system (5) with $\epsilon = 0$, is generated by the family of operators to which the monodromies accumulate when $\epsilon \rightarrow 0$.*

Remark 6. What we have essentially done here is that we have replaced the divergent term $e^{-2\pi i \frac{\lambda^{(0)}(\epsilon)}{(\epsilon)}}$ in the formal monodromy matrix $N_0(\epsilon)$ by an independent parameter $\kappa = e^{-2\pi i \frac{\lambda^{(0)}(\epsilon_0)}{(\epsilon_0)}}$ coming from the exponential torus.

3 The character variety of P_{VI} and the monodromy action on it

In this section we recall the usual approach to P_{VI} through isomonodromic deformations of 2×2 traceless linear systems with four Fuchsian singularities on \mathbb{CP}^1 , the Riemann-Hilbert correspondence between Fuchsian systems and their monodromy representations, the Fricke formulas for the character variety of P_{VI} and for the modular group action on it. The main references for this part are the articles of Iwasaki [Iwa03] and of Inaba, Iwasaki and Saito [IIS06].

Notation 7. A triple of indices (i, j, k) will always denote a permutation of $(0, t, 1)$, and (i, j, k, l) will denote a permutation of $(0, t, 1, \infty)$.

3.1 Isomonodromic deformations of Garnier systems and the Riemann-Hilbert correspondence

The sixth Painlevé equation $P_{VI}(\vartheta)$ with a parameter $\vartheta = (\vartheta_0, \vartheta_t, \vartheta_1, \vartheta_\infty) \in \mathbb{C}^4$ governs isomonodromic deformations of Garnier systems (traceless 2×2 linear systems) with four Fuchsian singularities on \mathbb{CP}^1

$$\frac{d\phi}{dx} = \left[\frac{A_0(t)}{x} + \frac{A_t(t)}{x-t} + \frac{A_1(t)}{x-1} \right] \phi \quad (9)$$

with the residue matrices $A_l \in \mathfrak{sl}_2(\mathbb{C})$ having $\pm \frac{\vartheta_l}{2}$ as eigenvalues. In general (if each A_l is semi-simple and the system is irreducible), one can write

$$A_i = \begin{pmatrix} z_i + \frac{\vartheta_i}{2} & -u_i z_i \\ \frac{z_i + \vartheta_i}{u_i} & -z_i - \frac{\vartheta_i}{2} \end{pmatrix}, \quad i = 0, t, 1, \quad -A_0 - A_t - A_1 = A_\infty = \begin{pmatrix} \frac{\vartheta_\infty}{2} & \\ & -\frac{\vartheta_\infty}{2} \end{pmatrix}.$$

The isomonodromicity of such system is expressed by the Schlesinger equations

$$\frac{dA_0}{dt} = \frac{[A_t, A_0]}{t}, \quad \frac{dA_t}{dt} = \frac{[A_0, A_t]}{t} + \frac{[A_1, A_t]}{t-1}, \quad \frac{dA_1}{dt} = \frac{[A_t, A_1]}{t-1}, \quad (10)$$

corresponding to the integrability conditions on the logarithmic connection in variables x, t

$$\nabla(x, t) = d - \left[A_0(t) d \log(x) + A_t(t) d \log(x-t) + A_1(t) d \log(x-1) \right]$$

on the rank 2 trivial vector bundle. Denoting $A(x, t) = (a_{ij}(x, t))$ the matrix of the system (9), then, if the system is irreducible, the 1-form $a_{12}(x, t) dx$ is non-null, so it must have a unique zero at some point $x = q(t)$. Denoting $p(t) = a_{11}(q, t) + \frac{\partial_0}{2q} + \frac{\partial_t}{2(q-t)} + \frac{\partial_1}{2(q-1)}$, the Schlesinger equations (10) are equivalent to the Hamiltonian system of P_{VI} (1) [Oka80, JM81].³

Choosing a germ of a fundamental matrix solution $\Phi(x, t)$ of the system, one has a monodromy representation (anti-homomorphism)

$$\rho : \pi_1(\mathbb{CP}^1 \setminus \{0, t, 1, \infty\}, x_0) \rightarrow \mathrm{SL}_2(\mathbb{C}),$$

such that the analytic continuation $\Phi(x, t)$ along a path γ defines another fundamental matrix solution $\Phi(x, t)\rho(\gamma)$. The conjugation class of such monodromy representation by $\mathrm{SL}_2(\mathbb{C})$ is independent of the choice of Φ . The word *isomonodromic* means that the conjugation class of monodromy is locally constant with respect to t , or equivalently that there exists a fundamental matrix solution $\Phi(x, t)$ whose actual monodromy is locally independent of t [Bol97].

The Riemann–Hilbert correspondence is given by the monodromy map between the space of linear systems (9) with prescribed poles and local eigenvalue data $\pm \frac{\partial_l}{2}$, modulo conjugation in $\mathrm{SL}_2(\mathbb{C})$ on one side, to the space of monodromy representations with prescribed local exponents $e_l, \frac{1}{e_l}$

$$e_l := e^{\pi \partial_l}, \quad l \in \{0, t, 1, \infty\}, \quad (11)$$

modulo conjugation in $\mathrm{SL}_2(\mathbb{C})$, on the other side (see [IIS06] for much more precise setting of the correspondence). Therefore it can be also translated as a correspondence between solutions of P_{VI} and equivalence classes of monodromy representations, between the Painlevé flow (1) on the moduli space of linear systems (9) and the locally constant “isomonodromic” flow on the moduli space of monodromy representations.

3.2 The character variety of P_{VI}

Given a representation

$$\rho : \pi_1(\mathbb{CP}^1 \setminus \{0, t, 1, \infty\}, x_0) \rightarrow \mathrm{SL}_2(\mathbb{C}),$$

let $\gamma_0, \gamma_t, \gamma_1, \gamma_\infty$ be simple loops in the x -space around $0, t, 1, \infty$ respectively such that $\gamma_0 \gamma_t \gamma_1 \gamma_\infty = \mathrm{id}$, and $M_l = \rho(\gamma_l)$ the corresponding monodromy matrices

$$M_\infty M_1 M_t M_0 = I.$$

³The change of variable $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_{11} & a_{12} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \frac{d\phi_1}{dx} \end{pmatrix}$ transforms the system (9) to a second order equation for ϕ_1 , which has an additional apparent singularity at the point $x = q(t)$ at which the above transformation is singular. The equation P_{VI} controls the position of this apparent singularity in the isomonodromic family, which was the original approach of Fuchs [Fu07]. There are many other ways to convert the system (9) to a single second order equation (every choice of a cyclic vector for the matrix A will do, and almost all vectors are cyclic), correspondingly there are many ways to define $q(t)$ in terms the matrix A .

The conjugacy class of an irreducible monodromy representation is completely determined by its trace coordinates by a theorem of Fricke, Klein and Vogt. These coordinates are given by the four parameters

$$a_l = \text{tr}(M_l) = e_l + \frac{1}{e_l} = 2 \cos(\pi \vartheta_l), \quad l = 0, t, 1, \infty, \quad (12)$$

and the three variables

$$X_0 = \text{tr}(M_1 M_t), \quad X_t = \text{tr}(M_0 M_1), \quad X_1 = \text{tr}(M_t M_0), \quad (13)$$

satisfying the *Fricke relation*

$$F(X, a) := X_0 X_t X_1 + X_0^2 + X_t^2 + X_1^2 - \theta_0 X_0 - \theta_t X_t - \theta_1 X_1 + \theta_\infty = 0, \quad (14)$$

where

$$\theta_i = a_i a_\infty + a_j a_k, \quad \text{for } i = 0, t, 1, \quad \text{and} \quad \theta_\infty = a_0 a_t a_1 a_\infty + a_0^2 + a_t^2 + a_1^2 + a_\infty^2 - 4.$$

Definition 8. We call “the character variety of P_{VI} ” the complex surface

$$\mathcal{S}_{VI}(a) = \{X \in \mathbb{C}^3 : F(X, a) = 0\}. \quad (15)$$

In this setting, the Riemann–Hilbert correspondence can be seen as a map between the Hamiltonian flow of the Painlevé system on one side and a locally constant Hamiltonian flow on the character variety on the other side. Under this correspondence the Okamoto space of initial conditions $\mathcal{M}_{VI,t}(\vartheta)$ is a minimal resolution of singularities of $\mathcal{S}_{VI}(a)$ [IIS06].

The character variety $\mathcal{S}_{VI}(a)$ is equipped with a natural algebraic symplectic form given by the *Poincaré residue*

$$\omega = \frac{dX_0 \wedge dX_t}{F_{X_1}} = \frac{dX_t \wedge dX_1}{F_{X_0}} = \frac{dX_1 \wedge dX_0}{F_{X_t}}, \quad (16)$$

where

$$F_{X_i} = \frac{dF}{dX_i} = X_j X_k + 2X_i - \theta_i.$$

3.3 Lines and singularities of $\mathcal{S}_{VI}(a)$

Proposition 9 (Lines of $\mathcal{S}_{VI}(a)$). *The Fricke polynomial $F(X, a)$ (14) can be decomposed as*

$$\begin{aligned} F(X, a) &= (X_k - \frac{e_i}{e_j} - \frac{e_j}{e_i})(F_{X_k} - X_k + \frac{e_i}{e_j} + \frac{e_j}{e_i}) \\ &\quad - \frac{1}{e_i e_j}(e_i X_i + e_j X_j - a_\infty - e_i e_j a_k)(e_i X_j + e_j X_i - a_k - e_i e_j a_\infty), \\ &= (X_k - e_i e_j - \frac{1}{e_i e_j})(F_{X_k} - X_k + e_i e_j + \frac{1}{e_i e_j}) \\ &\quad - \frac{1}{e_i e_j}(e_i e_j X_i + X_j - e_j a_\infty - e_i a_k)(e_i e_j X_j + X_i - e_j a_k - e_i a_\infty), \\ &= (X_k - \frac{e_k}{e_\infty} - \frac{e_\infty}{e_k})(F_{X_k} - X_k + \frac{e_k}{e_\infty} + \frac{e_\infty}{e_k}) \\ &\quad - \frac{1}{e_k e_\infty}(e_\infty X_i + e_k X_j - a_i - e_k e_\infty a_j)(e_k X_i + e_\infty X_j - a_j - e_k e_\infty a_i), \\ &= (X_k - e_k e_\infty - \frac{1}{e_k e_\infty})(F_{X_k} - X_k + e_k e_\infty + \frac{1}{e_k e_\infty}) \\ &\quad - \frac{1}{e_k e_\infty}(X_j + e_k e_\infty X_i - e_k a_i - e_\infty a_j)(X_i + e_k e_\infty X_j - e_k a_j - e_\infty a_i). \end{aligned}$$

In particular, the following 24 lines are contained in $\mathcal{S}_{VI}(a)$:

$$\begin{aligned}
&\{X_k = \frac{e_i}{e_j} + \frac{e_j}{e_i}, \quad e_i X_i + e_j X_j = a_\infty + e_i e_j a_k\}, \\
&\{X_k = \frac{e_i}{e_j} + \frac{e_j}{e_i}, \quad e_i X_j + e_j X_i = a_k + e_i e_j a_\infty\}, \\
&\{X_k = e_i e_j + \frac{1}{e_i e_j}, \quad X_i + e_i e_j X_j = e_j a_k + e_i a_\infty\}, \\
&\{X_k = e_i e_j + \frac{1}{e_i e_j}, \quad X_j + e_i e_j X_i = e_j a_\infty + e_i a_k\}, \\
&\{X_k = \frac{e_k}{e_\infty} + \frac{e_\infty}{e_k}, \quad e_\infty X_i + e_k X_j = a_i + e_k e_\infty a_j\}, \\
&\{X_k = \frac{e_k}{e_\infty} + \frac{e_\infty}{e_k}, \quad e_k X_i + e_\infty X_j = a_j + e_k e_\infty a_i\}, \\
&\{X_k = e_k e_\infty + \frac{1}{e_k e_\infty}, \quad X_i + e_k e_\infty X_j = e_k a_j + e_\infty a_i\}, \\
&\{X_k = e_k e_\infty + \frac{1}{e_k e_\infty}, \quad X_j + e_k e_\infty X_i = e_k a_i + e_\infty a_j\}.
\end{aligned}$$

The projective completion of $\mathcal{S}_{VI}(a)$ in \mathbb{CP}^3 contains 3 additional lines at infinity, giving the total of 27 lines provided by the classical Cayley-Salmon theorem. They are all distinct if and only if $\mathcal{S}_{VI}(a)$ is nonsingular.

Singular points of $\mathcal{S}_{VI}(a)$. The surface $\mathcal{S}_{VI}(a)$ is simply connected (cf. [CL09]), and it may or may not have singular points depending on a , but it never has more than 4 singular points [Obl04]. The singularities that appear correspond to unstable monodromy representations, which are of two kinds:

- Either $M_l = \pm I$ for some $l \in \{0, t, 1, \infty\}$, hence $e_l = \pm 1$.
If $l = i \in \{0, t, 1\}$, then $a_i = \pm 2$, $X_i = \pm a_\infty$, $X_j = \pm a_k$, $X_k = \pm a_j$.
If $l = \infty$, then $a_\infty = \pm 2$, $X_i = \pm a_i$, $i = 0, t, 1$.
- Or the representation is reducible, in which case $M_i = \begin{pmatrix} e_i^{\delta_i} & * \\ 0 & e_i^{-\delta_i} \end{pmatrix}$ for some triple of signs $(\delta_0, \delta_t, \delta_1) \in \{\pm 1\}^3$, and

$$e_0^{\delta_0} e_t^{\delta_t} e_1^{\delta_1} e_\infty = 1, \quad X_i = e_j^{\delta_j} e_k^{\delta_k} + e_j^{-\delta_j} e_k^{-\delta_k}.$$

The surface is therefore singular if and only if

$$\prod_{l \in \{0, t, 1, \infty\}} (a_l^2 - 4) \cdot w(a) = 0,$$

where

$$\begin{aligned}
w(a) &:= (a_0 + a_t + a_1 + a_\infty)(a_0 + a_\infty - a_t - a_1)(a_t + a_\infty - a_1 - a_0)(a_1 + a_\infty - a_0 - a_t) - \\
&\quad - (a_0 a_\infty - a_t a_1)(a_t a_\infty - a_1 a_0)(a_1 a_\infty - a_0 a_t) \\
&= \frac{(e_0 e_t e_1 e_\infty - 1)(e_0 e_t - e_1 e_\infty)(e_t e_1 - e_0 e_\infty)(e_1 e_0 - e_t e_\infty)}{(e_0 e_t e_1 e_\infty)^4} \prod_{l \in \{0, t, 1, \infty\}} (e_l - e_i e_j e_k),
\end{aligned}$$

see [Iwa02]. All the singularities of the projective completion of $\mathcal{S}_{VI}(a)$ are contained in its finite part, where they are situated on the intersection of 6 or more lines. Let $\mathcal{S}_{VI}^\circ(a) = \mathcal{S}_{VI}(a) \setminus \text{Sing}(\mathcal{S}_{VI}(a))$ be the smooth locus.

3.4 The modular action on $\mathcal{S}_{VI}(a)$

The non-linear *monodromy action* on the space of $\text{SL}_2(\mathbb{C})$ -monodromy representations, is given by the action of moving t along loops in $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ while keeping the

representation constant. When t returns to the initial position t_0 , the loops generating $\pi_1(\mathbb{CP}^1 \setminus \{0, t, 1, \infty\}, x_0)$ will not be the same as before. It induces an automorphism of the fundamental group through which it acts on the space of monodromy representations. The movement of t can be also seen as an action of the pure-braid group \mathcal{P}_3 on three strands $(0, t, 1)$, generated by the pure braids $\beta_{0t}^2, \beta_{t1}^2 \in \mathcal{P}_3$ (Figure 3).

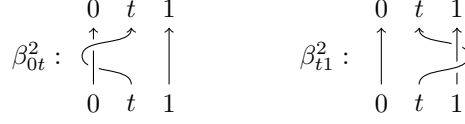


Figure 3: Elementary pure braids.

One can also consider the action of the whole braid group \mathcal{B}_3 , generated by the braids β_{0t}, β_{t1} (Figure 4), where the action of β_{ij} on the three points $(0, t, 1)$ (which are for the moment considered as movable), is a composition of two commuting actions

- 1) turning the two points i, j , around each other by a half turn,
- 2) swapping their names $i \leftrightarrow j$.

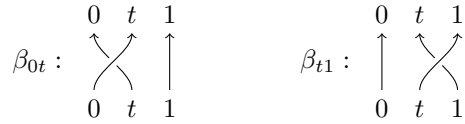


Figure 4: Elementary braids.

This action was described by Iwasaki [Iwa03], see also [PR15].

Proposition 10 (Iwasaki [Iwa03]).

1) The braid group on three strands \mathcal{B}_3 , with the generators β_{0t}, β_{t1} , acts on x -space $\mathbb{CP}^1 \setminus \{0, t, 1, \infty\}$ as the mapping class group by fixing ∞ , therefore inducing an action on the fundamental group $\pi_1(\mathbb{CP}^1 \setminus \{0, t, 1, \infty\}, x_0)$ through which it acts on the monodromy representations:

$$\begin{aligned} \beta_{0t} : M_0 &\mapsto M_0^{-1} M_t M_0, & \beta_{t1} : M_0 &\mapsto M_0, \\ M_t &\mapsto M_0, & M_t &\mapsto M_t^{-1} M_1 M_t, \\ M_1 &\mapsto M_1, & M_1 &\mapsto M_t. \end{aligned}$$

The induced “**half-monodromy**” actions g_{0t}, g_{t1} on the character variety $\mathcal{S}_{VI}(a)$ are given by

$$\begin{aligned} g_{ij} : \quad e_i &\mapsto e_j, & X_i &\mapsto X_j, & F_{X_i} &\mapsto F_{X_j} - F_{X_i} X_k, \\ e_i &\mapsto e_i, & X_j &\mapsto X_i - F_{X_i}, & F_{X_j} &\mapsto -F_{X_i}, \\ e_k &\mapsto e_k, & X_k &\mapsto X_k, & F_{X_k} &\mapsto F_{X_k} - F_{X_i} X_j, \\ e_\infty &\mapsto e_\infty. \end{aligned}$$

They preserve the Fricke relation: $F \circ g_{ij} = F$, and therefore they preserve also the 2-form ω . They satisfy

$$g_{ij} = g_{ji}^{-1}, \quad g_{ij} \circ g_{jk} \circ g_{ij} = g_{jk} \circ g_{ij} \circ g_{jk}, \quad g_{ki} = g_{ji} \circ g_{jk} \circ g_{ij} = g_{jk} \circ g_{ij} \circ g_{kj},$$

for any permutation (i, j, k) of $(0, t, 1)$. The group

$$\Gamma = \langle g_{0t}, g_{t1} \rangle,$$

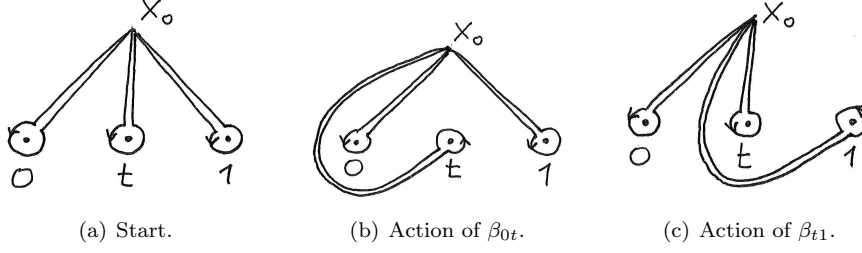


Figure 5: Action of the elementary braids $\beta_{0t}, \beta_{t1} \in \mathcal{B}_3$ on the fundamental group $\pi_1(\mathbb{CP}^1 \setminus \{0, t, 1, \infty\}, x_0)$.

generated by them is isomorphic to the **modular group** $\mathrm{PSL}_2(\mathbb{Z})$, with the standard generators

$$S = g_{t1} \circ g_{0t}^{\circ 2}, \quad T = g_{t0}, \quad \text{satisfying} \quad S^{\circ 2} = \mathrm{id} = (S \circ T)^{\circ 3}. \quad (17)$$

2) The action of the **monodromy group** of P_{VI} on the character variety is induced by the action of the pure braids $\beta_{0t}^2, \beta_{t1}^2 \in \mathcal{P}_3$ on the monodromy representations. It is isomorphic to the **principal congruence subgroup** of the modular group

$$\Gamma(2) = \langle g_{0t}^{\circ 2}, g_{t1}^{\circ 2} \rangle \subseteq \mathrm{Aut}(\mathcal{S}_{VI}(a)),$$

generated by

$$\begin{aligned} g_{0t}^{\circ 2} : \quad & X_0 \mapsto X_0 - F_{X_0}, & F_{X_0} &\mapsto -F_{X_0} - X_1 F_{X_t} + X_1^2 F_{X_0}, \\ & X_t \mapsto X_t - F_{X_t} + X_1 F_{X_0}, & F_{X_t} &\mapsto -F_{X_t} + X_1 F_{X_0}, \\ & X_1 \mapsto X_1, & F_{X_1} &\mapsto F_{X_1} - X_t F_{X_0} - X_0 F_{X_t} + F_{X_0} F_{X_t} + X_1 X_0 F_{X_0} - X_1 F_{X_0}^2, \\ g_{t1}^{\circ 2} : \quad & X_0 \mapsto X_0, & F_{X_0} &\mapsto F_{X_0} - X_1 F_{X_t} - X_t F_{X_1} + F_{X_t} F_{X_1} + X_0 X_t F_{X_t} - X_0 F_{X_t}^2, \\ & X_t \mapsto X_t - F_{X_t}, & F_{X_t} &\mapsto -F_{X_t} - X_0 F_{X_1} + X_0^2 F_{X_t}, \\ & X_1 \mapsto X_1 - F_{X_1} + X_0 F_{X_t}, & F_{X_1} &\mapsto -F_{X_1} + X_0 F_{X_t}, \end{aligned}$$

while preserving the parameters e_i . The fixed points of this action are exactly the singularities of \mathcal{S}_{VI} .

The modular group action on \mathcal{S}_{VI} has been studied in detail in [CL09].

Remark 11. *i)* The monodromy action of $\Gamma(2)$ on $\mathcal{S}_{VI}(a)$ corresponds through the Riemann-Hilbert correspondence to the non-linear monodromy action on the Painlevé foliation P_{VI} given by the Poincaré map. More precisely, the singular locus of $\mathcal{S}_{VI}(a)$ corresponds to so called Riccati solutions of P_{VI} , and the $\Gamma(2)$ -action on the smooth locus $\mathcal{S}_{VI}^\circ(a)$ represents faithfully the non-linear monodromy action on the non-Riccati locus of space of initial conditions $\mathcal{M}_{VI,t_0}(\vartheta)$ [IIS06].

ii) The half-monodromy actions map an equation P_{VI} with parameter to another equation P_{VI} with a permuted parameter. Therefore they make sense only if one considers the totality of all equations P_{VI} .

4 The confluence $P_{VI} \rightarrow P_V$ and the character varieties

4.1 Confluence of isomonodromic systems

The substitution

$$t \mapsto 1 + \epsilon, \quad \vartheta_t = \frac{1}{\epsilon}, \quad \vartheta_1 = -\frac{1}{\epsilon} + \tilde{\vartheta}_1 + 1, \quad (18)$$

in the system (9) with

$$z_1 = -\frac{\tilde{z}_1}{\epsilon t}, \quad z_t = \frac{\tilde{z}_1}{\epsilon t} - z_0 + \kappa_2, \quad \text{where} \quad \kappa_2 = -\frac{\vartheta_0 + \tilde{\vartheta}_1 + 1 + \vartheta_\infty}{2},$$

gives a parametric family (depending on the parameter ϵ) of isomonodromic deformations:

$$\frac{d\phi}{dx} = \left[\frac{A_0(t)}{x} + \frac{\tilde{A}_1^{(0)}(t) + (x-1-\epsilon t)\tilde{A}_1^{(1)}(t)}{(x-1)(x-1-\epsilon t)} \right] \phi, \quad (19)$$

where

$$\begin{aligned} \tilde{A}_1^{(0)} &= -\epsilon t A_1 = \begin{pmatrix} \tilde{z}_1 + \frac{t}{2} - \epsilon t \frac{1+\tilde{\vartheta}_1}{2} & -u_1 \tilde{z}_1 \\ \frac{\tilde{z}_1 + t - \epsilon t(1+\tilde{\vartheta}_1)}{u_1} & -\tilde{z}_1 - \frac{t}{2} + \epsilon t \frac{1+\tilde{\vartheta}_1}{2} \end{pmatrix}, \\ \tilde{A}_1^{(1)} &= A_t + A_1 = \begin{pmatrix} -z_0 - \frac{\vartheta_0 + \vartheta_\infty}{2} & u_0 z_0 \\ -\frac{z_0 + \vartheta_0}{u_0} & z_0 + \frac{\vartheta_0 + \vartheta_\infty}{2} \end{pmatrix}, \end{aligned}$$

which have well defined limits when $\epsilon \rightarrow 0$. The matrix $A_1^{(0)}$ has eigenvalues $\pm \frac{t - \epsilon t(1+\tilde{\vartheta}_1)}{2}$ with limit $\pm \frac{t}{2}$ when $\epsilon \rightarrow 0$.

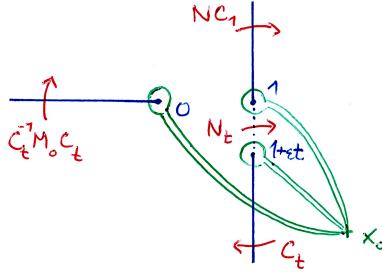


Figure 6: Monodromy of the system (19).

4.2 Confluence on character varieties

Now we can use the description of the linear confluence given in Section 2. Under the substitution (18), we have

$$e_t = e^{\frac{\pi i}{\epsilon}}, \quad e_1 = -e^{\pi i \tilde{\vartheta}_1 - \frac{\pi i}{\epsilon}}.$$

The monodromy matrices with respect to the canonical fundamental matrix solution described in Section 2 (Proposition 4) near the two confluent singularities, for $|\arg \frac{\epsilon}{t}| < \pi$, are of the following form (see Figure 4.1)

$$\begin{aligned} M_0 &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad M_t = C_t N_t = \begin{pmatrix} e_t & 0 \\ e_t c_t & \frac{1}{e_t} \end{pmatrix}, \quad M_1 = N_1 C_1 = \begin{pmatrix} e_1 & e_1 c_1 \\ 0 & \frac{1}{e_1} \end{pmatrix}, \\ M_\infty &= (M_t M_1 M_0)^{-1} = \begin{pmatrix} e_t e_1 (\beta c_t + \delta c_t c_1) + \frac{\delta}{e_t e_1} & -e_t e_1 (\beta + \delta c_1) \\ -e_t e_1 (\alpha c_t + \gamma c_t c_1) - \frac{\gamma}{e_t e_1} & e_t e_1 (\alpha + \gamma c_1) \end{pmatrix}, \end{aligned} \quad (20)$$

where $C_t = \begin{pmatrix} 1 & 0 \\ c_t & 1 \end{pmatrix}$, $C_1 = \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix}$, are the unfolded Stokes matrices, and $N_i = \begin{pmatrix} e_i & 0 \\ 0 & \frac{1}{e_i} \end{pmatrix}$, $i \in \{t, 1\}$, for $\epsilon \neq 0$, are the formal monodromies around the points $1 + \epsilon t$ and 1 . These monodromy matrices are subject to the conditions

$$\begin{aligned} \det(M_0) &= 1 = \alpha \delta - \beta \gamma, \\ \text{tr}(M_0) &= a_0 = \alpha + \delta, \\ \text{tr}(M_\infty) &= a_\infty = \frac{\delta}{e_t e_1} + e_t e_1 (\alpha + c_t \beta + c_1 \gamma + c_t c_1 \delta). \end{aligned}$$

This description is determined uniquely up to conjugation by diagonal matrices. The trace coordinates X_i (13) are given by

$$\begin{aligned} X_0 &= \text{tr}(M_1 M_t) = e_t e_1 + \frac{1}{e_t e_1} + e_t e_1 c_t c_1, \\ X_t &= \text{tr}(M_1 M_0) = e_1(\alpha + \gamma c_1) + \frac{\delta}{e_1}, \\ X_1 &= \text{tr}(M_0 M_t) = e_t(\alpha + \beta c_t) + \frac{\delta}{e_t}. \end{aligned} \quad (21)$$

Only the parameters

$$a_0 = 2 \cos(\pi \vartheta_0), \quad e_t e_1 = -e^{\pi i \tilde{\vartheta}_1} := -\tilde{e}_1, \quad a_\infty = 2 \cos(\pi \vartheta_\infty),$$

have well defined limits when $\epsilon \rightarrow 0$, while $e_t = e^{\frac{\pi i}{\epsilon}}$ and $e_1 = -e^{\pi i \tilde{\vartheta}_1 - \frac{\pi i}{\epsilon}}$ diverge. Therefore the coordinate X_0 passes well to the limit, but not X_t, X_1 which need be replaced by new coordinates (invariant with respect to diagonal conjugation).

Following [PS09], we choose as the new coordinates the low diagonal elements of M_∞ and M_0 :

$$\begin{aligned} W_t &= (M_\infty)_{22} = e_1 e_t(\alpha + \gamma c_1), \\ U_1 &= (M_0)_{22} = \delta. \end{aligned} \quad (22)$$

A substitution in the identity

$$e_t e_1 c_t c_1(\alpha \delta - \beta \gamma - 1) = 0,$$

gives the Fricke relation in the new coordinates

$$\tilde{F}(X_0, W_t, U_1, \tilde{a}) := X_0 W_t U_1 + W_t^2 + U_1^2 - \tilde{\theta}_0 X_0 - \tilde{\theta}_t W_t - \tilde{\theta}_1 U_1 + \tilde{\theta}_\infty = 0 \quad (23)$$

where $\tilde{\theta}_l$ are functions of the parameter

$$\tilde{a} = (a_0, -\tilde{e}_1, a_\infty),$$

independent of ϵ ,

$$\tilde{\theta}_0 = -\tilde{e}_1, \quad \tilde{\theta}_t = a_\infty - \tilde{e}_1 a_0, \quad \tilde{\theta}_1 = a_0 - \tilde{e}_1 a_\infty, \quad \tilde{\theta}_\infty = 1 - \tilde{e}_1 a_0 a_\infty + \tilde{e}_1^2. \quad (24)$$

The relation (23) is already known from [PS09, section 3.2].

Definition 12. The *wild character variety* of P_V is the complex surface

$$\mathcal{S}_V(\tilde{a}) = \{(X_0, W_t, U_1) \in \mathbb{C}^3 : \tilde{F}(X_0, W_t, U_1, \tilde{a}) = 0\}, \quad (25)$$

endowed with the algebraic symplectic form (28).

Remark 13. A very simple way to obtain the coordinates (X_0, W_t, U_1) on the wild character variety $\mathcal{S}_V(\tilde{a})$ is by taking the following limit:

- When $\epsilon \rightarrow 0$ radially with $\arg \epsilon \in]-\pi, 0[$, then $e_t \rightarrow 0$, $e_1 \rightarrow \infty$, and

$$\frac{a_1}{a_t} \rightarrow -\tilde{e}_1, \quad \frac{X_t}{a_t} \rightarrow \tilde{W}_t := -\tilde{e}_1(\alpha + \gamma c_1), \quad \frac{X_1}{a_t} \rightarrow \tilde{U}_1 := \delta.$$

$$\text{and } \frac{1}{a_t^2} F(X_0, X_t, X_1, a) \rightarrow \tilde{F}(X_0, W_t, U_1, \tilde{a}).$$

- When $\epsilon \rightarrow 0$ radially with $\arg \epsilon \in]0, \pi[$, then $e_t \rightarrow \infty$, $e_1 \rightarrow 0$, and

$$\frac{a_t}{a_1} \rightarrow -\tilde{e}_1, \quad \frac{X_t}{a_1} \rightarrow \tilde{U}_t := \delta, \quad \frac{X_1}{a_1} \rightarrow \tilde{W}_1 := -\tilde{e}_1(\alpha + \beta c_t),$$

$$\text{and } \frac{1}{a_1^2} F(X_0, X_t, X_1, a) \rightarrow \tilde{F}(X_0, U_t, W_1, \tilde{a}).$$

These two limit coordinates are related by the limit of the action of the half-monodromy operator g_{t1}

$$\bar{g}_{t1} : (X_0, W_t, U_1) \mapsto (X_0, U_t, W_1) = (X_0, U_1, W_t - \tilde{F}_{W_t}).$$

However, this limit description of the character variety of P_V , which has been more-less known (see for example [CMR15]), is too simple for our purpose. While the monodromy operator $g_{t1}^{\circ 2}$ has a well defined limit in this description, the monodromy operator $g_{0t}^{\circ 2}$ diverges as expected. It is precisely the confluence $\epsilon \rightarrow 0$ in the two omitted resonant directions $\pm \mathbb{R}^+$ that is interesting for us. Indeed, we want to be able to consider the limits along sequences $\{\epsilon_n\}_{n \in \pm \mathbb{N}}$ with $\frac{1}{\epsilon_n} = \frac{1}{\epsilon_0} + n$, along which e_t is constant, in order to obtain the generators of the wild monodromy.

The original variables (X_0, X_t, X_1) on the character variety $\mathcal{S}_{VI}(a)$ and the new variables (X_0, W_t, U_1) on the wild character variety $\mathcal{S}_V(\tilde{a})$ are related by the following rational transformations:

Theorem 14 ($\epsilon \neq 0$). *i) The affine varieties $\mathcal{S}_{VI}(a)$ and $\mathcal{S}_V(\tilde{a})$ are birationally equivalent through the change of variables*

$$\begin{aligned} \Phi : \mathcal{S}_V(\tilde{a}) &\rightarrow \mathcal{S}_{VI}(a) \\ (X_0, W_t, U_1, e) &\mapsto (X_0, X_t, X_1, e), \end{aligned}$$

given by

$$\begin{aligned} X_0 &= X_0, & F_{X_0} \circ \Phi &= \frac{U_1}{e_1} \left(\frac{\tilde{F}_{U_1}}{e_t} - \frac{\tilde{F}_{W_t}}{e_1} \right) - \frac{\tilde{F}_{X_0}}{e_t e_1} (X_0 - \frac{e_t}{e_1} - \frac{e_1}{e_t}) \\ X_t &= \frac{U_1}{e_1} + \frac{W_t}{e_t}, & F_{X_t} \circ \Phi &= \frac{\tilde{F}_{U_1}}{e_1} + \frac{\tilde{F}_{W_t}}{e_t} - \frac{X_0 \tilde{F}_{W_t}}{e_1} \\ X_1 &= \frac{U_1}{e_t} + \frac{W_t}{e_1} - \frac{\tilde{F}_{W_t}}{e_1}, & F_{X_1} \circ \Phi &= \frac{\tilde{F}_{U_1}}{e_t} - \frac{\tilde{F}_{W_t}}{e_1}, \end{aligned}$$

where

$$\begin{aligned} \tilde{F}_{X_0} &:= \frac{\partial \tilde{F}}{\partial X_0} = U_1 W_t - \tilde{\theta}_0, \\ \tilde{F}_{W_t} &:= \frac{\partial \tilde{F}}{\partial W_t} = X_0 U_1 + 2W_t - \tilde{\theta}_t, \\ \tilde{F}_{U_1} &:= \frac{\partial \tilde{F}}{\partial U_1} = X_0 W_t + 2U_1 - \tilde{\theta}_1. \end{aligned}$$

And the inverse map $\Phi^{\circ(-1)} : \mathcal{S}_{VI}(a) \dashrightarrow \mathcal{S}_V(\tilde{a})$ is given by

$$\begin{aligned} U_1 &= -\frac{e_t X_t + e_1 X_1 - \tilde{\theta}_t}{X_0 - \frac{e_t}{e_1} - \frac{e_1}{e_t}} = -e_t e_1 \frac{X_0 - \frac{e_t}{e_1} - \frac{e_1}{e_t} - F_{X_0}}{e_1 X_t + e_t X_1 - \tilde{\theta}_1}, & (\text{cf. Proposition 9}), \\ W_t &= -\frac{e_1 X_t + e_t X_1 - \tilde{\theta}_1 - e_t F_{X_1}}{X_0 - \frac{e_t}{e_1} - \frac{e_1}{e_t}} = e_t X_t - \frac{e_t}{e_1} U_1. \end{aligned} \tag{26}$$

which is singular on the line:

$$L_0(a) := \{X_0 = \frac{e_t}{e_1} + \frac{e_1}{e_t}, \quad e_1 X_t + e_t X_1 = \tilde{\theta}_1\}. \tag{27}$$

The two Fricke relations are related by

$$F \circ \Phi = \frac{-1}{e_t e_1} (X_0 - \frac{e_t}{e_1} - \frac{e_1}{e_t}) \cdot \tilde{F}.$$

The restriction

$$\Phi : \mathcal{S}_V(\tilde{a}) \rightarrow \mathcal{S}_{VI}(a) \setminus L_0(a)$$

is an isomorphism.

ii) The pull-back of the symplectic form ω (16) by Φ is a symplectic form on the smooth locus of $\mathcal{S}_V(\tilde{a})$ given by the Poincaré residue

$$\tilde{\omega} = \frac{dX_0 \wedge dW_t}{\tilde{F}_{U_1}} = \frac{dW_t \wedge dU_1}{\tilde{F}_{X_0}} = \frac{dU_1 \wedge dX_0}{\tilde{F}_{W_t}}. \quad (28)$$

Proof. i) Follows from (21) and (22).

ii) It can be verified by a direct calculation using the formulas of Proposition 14. \square

Remark 15. The singular line $L_0(a)$ on $\mathcal{S}_{VI}(a)$ corresponds to a “line at infinity” in the coordinates (X_0, W_t, U_1) . In fact, if one sets

$$(X_0, W_t, U_1) = (Y_0 + \frac{e_t}{e_1} + \frac{e_1}{e_t}, \frac{Y_t}{Y_0}, \frac{Y_1}{Y_0}),$$

then the image of point $X \in L_0(a)$ by $\Phi^{o(-1)}$ is given by:

$$(Y_0, Y_t, Y_1) = (0, e_t F_{X_1} - e_t X_t - e_1 X_1 + \tilde{\theta}_1, -e_t X_t - e_1 X_1 + \tilde{\theta}_t),$$

which satisfies

$$Y_0 = 0, \frac{Y_t}{e_t} + \frac{Y_1}{e_1} = 0,$$

and we may want to add this line to the surface $\{\tilde{F}(X_0, W_t, U_1, \tilde{a}) = 0\}$. This can be done in the following way:

Let (Y_0, Y_t, Y_1) be the coordinates on \mathbb{C}^3 and let $\tilde{\mathbb{C}}^3$ be its blowup at the origin given by the quotient

$$\tilde{\mathbb{C}}^3 = (\mathbb{C}^3 \setminus \{0\}) \times \mathbb{C} / \sim,$$

where $(Z_0 : Z_t : Z_1 : Z_\infty) \sim (\frac{Z_0}{\lambda} : \frac{Z_t}{\lambda} : \frac{Z_1}{\lambda} : \lambda Z_\infty)$ for any $\lambda \in \mathbb{C}^*$, $(Z_0 : Z_t : Z_1 : Z_\infty) \in (\mathbb{C}^3 \setminus \{0\}) \times \mathbb{C}$. Setting $Y_i = Z_i Z_\infty$ leads to the usual definition

$$\tilde{\mathbb{C}}^3 \simeq \{(Y_0, Y_t, Y_1; Z_0 : Z_t : Z_1) \in \mathbb{C}^3 \times \mathbb{CP}^2 : Y_i Z_j = Y_j Z_i\}.$$

The local coordinates on $\tilde{\mathbb{C}}^3$ are given by

$$\begin{aligned} (Z_0 : Z_t : Z_1 : Z_\infty) &\sim (Y_0 : Y_t : Y_1; 1) \quad \text{if } Z_\infty \neq 0, \\ &\sim (1 : \frac{Z_t}{Z_0} : \frac{Z_1}{Z_0}; Y_0) \quad \text{if } Z_0 \neq 0, \\ &\sim (\frac{Z_0}{Z_t} : 1 : \frac{Z_1}{Z_t}; Y_t) \quad \text{if } Z_t \neq 0, \\ &\sim (\frac{Z_0}{Z_1} : \frac{Z_t}{Z_1} : 1; Y_1) \quad \text{if } Z_1 \neq 0, \end{aligned}$$

and we identify

$$(X_0 - \frac{e_t}{e_1} - \frac{e_1}{e_t}, W_t, U_1) = (Z_0 Z_\infty, \frac{Z_t}{Z_0}, \frac{Z_1}{Z_0}) = (Y_0, \frac{Y_t}{Y_0}, \frac{Y_1}{Y_0}).$$

Then the intersection of the closure of the surface $\{\tilde{F}(X_0, W_t, U_1, \tilde{a}) = 0\}$ in $\tilde{\mathbb{C}}^3$ with the “plane at infinity” $\{Z_0 = 0\}$ consists of two lines (disjoint if $e_t^2 \neq e_1^2$):

$$Z_0^2 \tilde{F}(\frac{e_t}{e_1} + \frac{e_1}{e_t}, \frac{Z_t}{Z_0}, \frac{Z_1}{Z_0}) = (e_t Z_t + e_1 Z_1)(\frac{Z_t}{e_t} + \frac{Z_1}{e_1}) = 0.$$

The singular line $L_0(a)$ in $\mathcal{S}_{VI}(a)$ is mapped by $\Phi^{o(-1)}$ to the line $\tilde{L}_0(\tilde{a}, e_t) = \{Z_0 = 0, \frac{Z_t}{e_t} + \frac{Z_1}{e_1} = 0\}$,

$$X \in L_0(a) \mapsto (0 : -\frac{e_t}{e_1} : 1; \tilde{\theta}_t - e_t X_t - e_1 X_1) \in \tilde{\mathbb{C}}^3.$$

We define

$$\tilde{\mathcal{S}}_{VI}(\tilde{a}, e_t) = \{(X_0, W_t, U_1) \in \mathbb{C}^3 : \tilde{F}(X_0, W_t, U_1, \tilde{a}) = 0\} \cup \tilde{L}_0(\tilde{a}, e_t) \subseteq \tilde{\mathbb{C}}^3. \quad (29)$$

If $e_t^2 \neq e_1^2$, then the transformation Φ (26) is an isomorphisms between $\mathcal{S}_{VI}(a)$ and $\tilde{\mathcal{S}}_{VI}(\tilde{a}, e_t)$. Indeed, if $e_t^2 \neq e_1^2$, then $e_t X_t + e_1 X_1 - \tilde{\theta}_t$ is a coordinate on the line L_0 , whose image is therefore \tilde{L}_0 . Hence the map $\Phi^{o(-1)}$ extends to a smooth bijection $\mathcal{S}_{VI}(a) \rightarrow \tilde{\mathcal{S}}_{VI}(\tilde{a}, e_t)$.

Remark 16 ($\epsilon \neq 0$). In the trace coordinates (12) on the space of monodromy representations, the eigenvalues e_i and $\frac{1}{e_i}$ are interchangeable. On the other hand in the above description of the monodromy data (20) this is no longer true for $i = t, 1$. While most monodromy representations can be conjugated so that M_t is lower triangular and M_1 is upper triangular, one cannot always assure that e_t, e_1 are on the $(1, 1)$ -position. Those monodromy representations for which this is not possible correspond exactly to the points of the line at infinity $\tilde{L}_0(\tilde{a}, e_t)$.

4.3 Lines and singularities of $\mathcal{S}_V(\tilde{a})$

Proposition 17 (Lines of $\mathcal{S}_V(\tilde{a})$). *The polynomial \tilde{F} (23) can be decomposed as*

$$\begin{aligned}
\tilde{F}(X_0, W_t, U_1, \tilde{a}) &= \\
&= (X_0 + \tilde{e}_1 + \frac{1}{\tilde{e}_1})\tilde{F}_{X_0} - \tilde{e}_1(W_t - \frac{U_1}{\tilde{e}_1} - a_\infty)(U_1 - \frac{W_t}{\tilde{e}_1} - a_0), \\
&= (X_0 - e_0e_\infty - \frac{1}{e_0e_\infty})\tilde{F}_{X_0} + (e_0W_t + \frac{U_1}{e_\infty} + \tilde{e}_1 - \frac{e_0}{e_\infty})(\frac{W_t}{e_0} + \tilde{e}_\infty U_1 + \tilde{e}_1 - \frac{e_\infty}{e_0}), \\
&= (X_0 - \frac{e_0}{e_\infty} - \frac{e_\infty}{e_0})\tilde{F}_{X_0} + (e_0W_t + e_\infty U_1 + \tilde{e}_1 - e_0e_\infty)(\frac{W_t}{e_0} + \frac{U_1}{\tilde{e}_\infty} + \tilde{e}_1 - \frac{1}{e_0e_\infty}), \\
&= (W_t - e_\infty)(\tilde{F}_{W_t} - W_t + e_\infty) + (U_1 + \frac{\tilde{e}_1}{e_\infty})(e_\infty X_0 + U_1 + \tilde{e}_1e_\infty - a_0), \\
&= (W_t - \frac{1}{e_\infty})(\tilde{F}_{W_t} - W_t + \frac{1}{e_\infty}) + (U_1 + \tilde{e}_1e_\infty)(\frac{X_0}{e_\infty} + U_1 + \frac{\tilde{e}_1}{e_\infty} - a_0), \\
&= (W_t + \tilde{e}_1e_0)(\tilde{F}_{W_t} - W_t - \tilde{e}_1e_0) + (U_1 - \frac{1}{e_0})(-\tilde{e}_1e_0X_0 + U_1 - e_0 + \tilde{e}_1a_\infty), \\
&= (W_t + \frac{\tilde{e}_1}{e_0})(\tilde{F}_{W_t} - W_t - \frac{\tilde{e}_1}{e_0}) + (U_1 - e_0)(-\frac{\tilde{e}_1}{e_0}X_0 + U_1 - \frac{1}{e_0} + \tilde{e}_1a_\infty), \\
&= (U_1 - e_0)(\tilde{F}_{U_1} - U_1 + e_0) + (W_t + \frac{\tilde{e}_1}{e_0})(e_0X_0 + W_t + \tilde{e}_1e_0 - a_\infty), \\
&= (U_1 - \frac{1}{e_0})(\tilde{F}_{U_1} - U_1 + \frac{1}{e_0}) + (W_t + \tilde{e}_1e_0)(\frac{X_0}{e_0} + W_t + \frac{\tilde{e}_1}{e_0} - a_\infty), \\
&= (U_1 + \tilde{e}_1e_\infty)(\tilde{F}_{U_1} - U_1 - \tilde{e}_1e_\infty) + (W_t - \frac{1}{e_\infty})(-\tilde{e}_1e_\infty X_0 + W_t - e_\infty + \tilde{e}_1a_0), \\
&= (U_1 + \frac{\tilde{e}_1}{e_\infty})(\tilde{F}_{U_1} - U_1 - \frac{\tilde{e}_1}{e_\infty}) + (W_t - e_\infty)(-\frac{\tilde{e}_1}{e_\infty}X_0 + W_t - \frac{1}{e_\infty} + \tilde{e}_1a_0),
\end{aligned}$$

defining thus 22 lines on $\mathcal{S}_V(\tilde{a})$.

The projective completion of $\mathcal{S}_V(\tilde{a})$ in \mathbb{CP}^3 contains 3 additional lines at infinity.

Proposition 18 (Singular points of $\mathcal{S}_V(\tilde{a})$). *The affine cubic variety $\mathcal{S}_V(\tilde{a})$ has singular points if and only if*

$$(a_0^2 - 4)(a_\infty^2 - 4)\tilde{w}(\tilde{a}) = 0,$$

where

$$\begin{aligned}
\tilde{w}(\tilde{a}) &= (a_0^2 + \tilde{a}_1^2 + a_\infty^2 + a_0\tilde{a}_1a_\infty - 4), \quad \text{with } \tilde{a}_1 = \tilde{e}_1 + \frac{1}{\tilde{e}_1}, \\
&= \frac{(e_0e_te_1e_\infty - 1)(e_0e_te_1 - e_\infty)(e_te_1e_\infty - e_0)(e_te_1 - e_0e_\infty)}{(e_0e_te_1e_\infty)^2},
\end{aligned}$$

(cf. [PS09, section 3.2.2]). The corresponding possible singularities are the following:

- if $a_0 = \pm 2$: $X_0 = \pm a_\infty$, $W_t = \mp \tilde{e}_1$, $U_1 = \pm 1$,
- if $a_\infty = \pm 2$: $X_0 = \pm a_0$, $W_t = \pm 1$, $U_1 = \mp \tilde{e}_1$,
- if $-\tilde{e}_1e_0e_\infty = 1$: $X_0 = -\tilde{e}_1 - \frac{1}{\tilde{e}_1}$, $W_t = \frac{1}{e_\infty}$, $U_1 = \frac{1}{e_0}$,
- if $-\tilde{e}_1 = e_0e_\infty$: $X_0 = -\tilde{e}_1 - \frac{1}{\tilde{e}_1}$, $W_t = e_\infty$, $U_1 = e_0$.

Setting $(X_0, W_t, U_1) = (\frac{x_0}{v_\infty}, \frac{w_t}{v_\infty}, \frac{u_1}{v_\infty})$, the projective completion of $\mathcal{S}_V(\tilde{a})$ in \mathbb{CP}^3 has also a singularity at the point $(x_0 : w_t : u_1 : v_\infty) = (1 : 0 : 0 : 0)$ for any value of the parameters.

Proof. The surface $\mathcal{S}_V(\tilde{a})$ is isomorphic to $\mathcal{S}_{VI}(a) \setminus L_0(a)$, we can therefore use the description of the singular points of $\mathcal{S}_{VI}(a)$ given in Section 3.3. \square

4.4 The wild monodromy action on $\mathcal{S}_V(\tilde{a})$

The only monodromy actions on $\mathcal{S}_V(\tilde{a})$ that survive the confluence are those generated by $\tilde{g}_{i1}^{\circ 2}$. As was motivated in the introduction, we may instead consider limits along sequences $(\epsilon_n)_{n \in \mathbb{Z}}$, $\frac{1}{\epsilon_n} = \frac{1}{\epsilon_0} + n$ for which the divergent parameter $e_t = e^{\frac{\pi i}{\epsilon}}$ stays constant. This amounts to replacing e_t^2 by a new independent parameter $\nu = e^{\frac{2\pi i}{\epsilon_0}}$

$$e_t e_1 = -\tilde{e}_1, \quad e_t^2 = \nu \in \mathbb{C}^*.$$

For $\epsilon \neq 0$, let Φ be the corresponding map (7). In the coordinates (X_0, W_t, U_1) the pure braid group \mathcal{P}_3 (cf. Proposition 10) acts on $\mathcal{S}_V(\tilde{a})$ by the monodromy actions

$$\tilde{g}_{ij}^{\circ 2} = \Phi^{-1} \circ g_{ij}^{\circ 2} \circ \Phi,$$

preserving the Poincaré residue form $\tilde{\omega}$.

The *monodromy action*

$$\begin{aligned} \tilde{g}_{t1}^{\circ 2} : \quad & X_0 \mapsto X_0, & \tilde{F}_{X_0} &\mapsto \tilde{F}_{X_0} - W_t \tilde{F}_{U_1} - U_1 \tilde{F}_{W_t} + \tilde{F}_{W_t} \tilde{F}_{U_1} + X_0 W_t \tilde{F}_{W_t} - X_0 \tilde{F}_{W_t}^2, \\ & W_t \mapsto W_t - \tilde{F}_{W_t}, & \tilde{F}_{W_t} &\mapsto -\tilde{F}_{W_t} - X_0 \tilde{F}_{U_1} + X_0^2 \tilde{F}_{W_t}, \\ & U_1 \mapsto U_1 - \tilde{F}_{U_1} + X_0 \tilde{F}_{W_t}, & \tilde{F}_{U_1} &\mapsto -\tilde{F}_{U_1} + X_0 \tilde{F}_{W_t}, \\ \\ \tilde{g}_{1t}^{\circ 2} : \quad & X_0 \mapsto X_0, & \tilde{F}_{X_0} &\mapsto \tilde{F}_{X_0} - W_t \tilde{F}_{U_1} - U_1 \tilde{F}_{W_t} + \tilde{F}_{W_t} \tilde{F}_{U_1} + X_0 U_1 \tilde{F}_{U_1} - X_0 \tilde{F}_{U_1}^2, \\ & W_t \mapsto W_t - \tilde{F}_{W_t} + X_0 \tilde{F}_{U_1}, & \tilde{F}_{W_t} &\mapsto -\tilde{F}_{W_t} + X_0 \tilde{F}_{U_1}, \\ & U_1 \mapsto U_1 - \tilde{F}_{U_1}, & \tilde{F}_{U_1} &\mapsto -\tilde{F}_{U_1} - X_0 \tilde{F}_{W_t} + X_0^2 \tilde{F}_{U_1}, \end{aligned}$$

is independent of e_t and preserves the Fricke relation:

$$\tilde{F} \circ \tilde{g}_{t1}^2 = \tilde{F}.$$

The formulas for the *wild monodromy action* $\tilde{g}_{0t}^{\circ 2} = \tilde{g}_{0t}^{\circ 2}(\nu)$ and its inverse $\tilde{g}_{t0}^{\circ 2} = \tilde{g}_{t0}^{\circ 2}(\nu)$ are too complex to be written here. They depend only on $\nu = e_t^2$: indeed, the action of g_{0t} on $(X_0, \frac{X_t}{e_t}, \frac{X_1}{e_1})$ depends only on $e_t e_1$ and $\frac{e_t}{e_1}$, and so does the transformation $(X_0, W_t, U_1) \mapsto (X_0, \frac{X_t}{e_t}, \frac{X_1}{e_1})$ and its inverse. Their action change the Fricke relation by a factor:

$$\tilde{F} \circ \tilde{g}_{0t}^{\circ 2}(\nu) = \frac{(X_0 + \frac{\tilde{e}_1}{\nu} + \frac{\nu}{\tilde{e}_1})}{(X_0 + \frac{\tilde{e}_1}{\nu} + \frac{\nu}{\tilde{e}_1} - F_{X_0} \circ \Phi)} \tilde{F}.$$

Theorem 19 (Wild monodromy action on $\mathcal{S}_V(\tilde{a})$). *The wild monodromy actions are those in the group generated by*

$$\langle \tilde{g}_{t1}^{\circ 2}, \tilde{g}_{0t}^{\circ 2}(\nu) \mid \nu \in \mathbb{C}^* \rangle,$$

where $\tilde{g}_{t1}^{\circ 2}$ is the monodromy operator (independent of ν) and $\tilde{g}_{t0}^{\circ 2}(\nu) = \tilde{g}_{0t}^{\circ -2}(\nu)$, $\tilde{g}_{t1}^{\circ 2} \circ \tilde{g}_{0t}^{\circ 2}(\nu)$ are wild monodromy operators, acting on $\mathcal{S}_V(\tilde{a})$ as $\tilde{\omega}$ -preserving birational transformations. For each ν fixed $\langle \tilde{g}_{t1}^{\circ 2}, \tilde{g}_{0t}^{\circ 2}(\nu) \rangle \simeq \Gamma(2)$.

Theorem 20 (Torus action). *The wild monodromy operators $\tilde{g}_{t0}^{\circ 2}(\nu)$, and $\tilde{g}_{t1}^{\circ 2} \circ \tilde{g}_{0t}^{\circ 2}(\nu)$, can be written as*

$$\tilde{g}_{t0}^{\circ 2}(\nu) = \tau_{U_1}(\nu) \circ \sigma, \quad \text{and} \quad \tilde{g}_{t1}^{\circ 2} \circ \tilde{g}_{0t}^{\circ 2}(\nu) = \tau_{W_t}(\nu)^{\circ -1} \circ \sigma' = \sigma' \circ \tau_{U_1}(\nu)^{\circ -1},$$

where

- σ, σ' are “Stokes operators” (independent of ν),
- $\tilde{g}_{t1}^{\circ 2} = \sigma' \circ \sigma$ is the monodromy operator,
- $\tau_{U_1}(\nu), \tau_{W_t}(\nu)$ are “torus actions”.

The vector field $(\dot{X}_0, \dot{W}_t, \dot{U}_1) = \nu \frac{d}{d\nu} \tau_{U_1}(\nu)$, which generates infinitesimally the action of $\tau_{U_1}(\nu)$, is a Hamiltonian vector field on $\mathcal{S}_V(\tilde{a})$ with respect to the symplectic form $\tilde{\omega}$ (28)⁴ and to the Hamiltonian function $H_{U_1} = \log(U_1)$, given by

$$\frac{\tilde{F}_{W_t}}{U_1} \partial_{X_0} - \frac{\tilde{F}_{X_0}}{U_1} \partial_{W_t}. \quad (30)$$

And the vector field $(\dot{X}_0, \dot{W}_t, \dot{U}_1) = \nu \frac{d}{d\nu} \tau_{W_t}(\nu)$, which generates infinitesimally the action of $\tau_{W_t}(\nu)$, is a Hamiltonian vector field on $\mathcal{S}_V(\tilde{a})$ with respect to the symplectic form $\tilde{\omega}$ (28) and the Hamiltonian function $H_{W_t} = \log(W_t)$, given by

$$-\frac{\tilde{F}_{U_1}}{W_t} \partial_{X_0} + \frac{\tilde{F}_{X_0}}{W_t} \partial_{U_1}. \quad (31)$$

Proof. By formal calculations in SageMath using the formulas of Proposition 10 and Proposition 14. One verifies that the vector field

$$\nu \frac{d}{d\nu} \tau_{U_1}(\nu) = \left(\nu \frac{d}{d\nu} \tilde{g}_{t0}^{\circ 2}(\nu) \right) \circ \tilde{g}_{0t}^{\circ 2}(\nu)$$

is given by (30) modulo \tilde{F} , and that the vector field

$$\nu \frac{d}{d\nu} \tau_{W_t}(\nu) = -\left(\nu \frac{d}{d\nu} \tilde{g}_{t1}^{\circ 2}(\nu) \circ \tilde{g}_{0t}^{\circ 2}(\nu) \right) \circ \tilde{g}_{1t}^{\circ 2}(\nu)$$

is given by (31) modulo \tilde{F} . \square

Remark 21. The action of the wild half-monodromy $\tilde{g}_{t1}(\nu) = \Phi^{-1} \circ g_{t1} \circ \Phi$ and its inverse $\tilde{g}_{1t}(\nu)$ on $\mathcal{S}_V(\tilde{a})$ are given by

$$\begin{aligned} \tilde{g}_{t1}(\nu) : \quad X_0 &\mapsto X_0, & \tilde{F}_{X_0} &\mapsto \tilde{F}_{X_0} - U_1 \tilde{F}_{W_t} - (U_1 + \frac{\nu}{\tilde{e}_1} W_t - \frac{\nu}{\tilde{e}_1} \tilde{F}_{W_t} - \frac{\nu}{\tilde{e}_1} \square) \square, \\ W_t &\mapsto W_t - \tilde{F}_{W_t} - \square, & \tilde{F}_{W_t} &\mapsto -\tilde{F}_{W_t} - \frac{\nu}{\tilde{e}_1} (X_0 + 2 \frac{\tilde{e}_1}{\nu}) \square, \\ U_1 &\mapsto U_1 - \frac{\nu}{\tilde{e}_1} \square, & \tilde{F}_{U_1} &\mapsto \tilde{F}_{U_1} - X_0 \tilde{F}_{W_t} - (X_0 + 2 \frac{\nu}{\tilde{e}_1}) \square, \\ \frac{\nu}{\tilde{e}_1} &\mapsto \frac{\tilde{e}_1}{\nu}, \end{aligned}$$

$$\text{where } \square := \frac{\tilde{F}_{U_1} - (X_0 + \frac{\tilde{e}_1}{\nu}) \tilde{F}_{W_t}}{X_0 + \frac{\tilde{e}_1}{\nu} + \frac{\nu}{\tilde{e}_1}}, \quad \square \mapsto \frac{\tilde{F}_{U_1} + \frac{\nu}{\tilde{e}_1} \tilde{F}_{W_t}}{X_0 + \frac{\tilde{e}_1}{\nu} + \frac{\nu}{\tilde{e}_1}} + \frac{\nu}{\tilde{e}_1} X_0 \square,$$

$$\begin{aligned} \tilde{g}_{1t}(\nu) : \quad X_0 &\mapsto X_0, & \tilde{F}_{X_0} &\mapsto \tilde{F}_{X_0} - W_t \tilde{F}_{U_1} - (W_t + \frac{\tilde{e}_1}{\nu} U_1 - \frac{\tilde{e}_1}{\nu} \tilde{F}_{U_1} - \frac{\tilde{e}_1}{\nu} \blacksquare) \blacksquare, \\ W_t &\mapsto W_t - \frac{\tilde{e}_1}{\nu} \blacksquare, & \tilde{F}_{W_t} &\mapsto \tilde{F}_{W_t} - X_0 \tilde{F}_{U_1} - (X_0 + 2 \frac{\tilde{e}_1}{\nu}) \blacksquare, \\ U_1 &\mapsto U_1 - \tilde{F}_{U_1} - \blacksquare, & \tilde{F}_{U_1} &\mapsto -\tilde{F}_{U_1} - \frac{\tilde{e}_1}{\nu} (X_0 + 2 \frac{\nu}{\tilde{e}_1}) \blacksquare, \\ \frac{\nu}{\tilde{e}_1} &\mapsto \frac{\tilde{e}_1}{\nu}, \end{aligned}$$

$$\text{where } \blacksquare := \frac{\tilde{F}_{W_t} - (X_0 + \frac{\nu}{\tilde{e}_1}) \tilde{F}_{U_1}}{X_0 + \frac{\tilde{e}_1}{\nu} + \frac{\nu}{\tilde{e}_1}}, \quad \blacksquare \mapsto \frac{\tilde{F}_{W_t} + \frac{\tilde{e}_1}{\nu} \tilde{F}_{U_1}}{X_0 + \frac{\tilde{e}_1}{\nu} + \frac{\nu}{\tilde{e}_1}} + \frac{\tilde{e}_1}{\nu} X_0 \blacksquare.$$

The second iterate $\tilde{g}_{t1}(\nu) \circ \tilde{g}_{1t}(\nu) = \tilde{g}_{t1}^{\circ 2}$ is the monodromy operator.

A different action on $\mathcal{S}_V(\tilde{a})$ whose square iterate is the monodromy operator $\tilde{g}_{t1}^{\circ 2}$ is obtained as the limit of the half-monodromy operator g_{t1} in the description of Remark 13

$$\begin{aligned} \bar{g}_{t1} : e_0 &\mapsto e_0, & X_0 &\mapsto X_0, & \tilde{F}_{X_0} &\mapsto \tilde{F}_{X_0} - \tilde{F}_{W_t} U_1, \\ \tilde{e}_1 &\mapsto \tilde{e}_1, & W_t &\mapsto U_t = U_1, & \tilde{F}_{W_t} &\mapsto \tilde{F}_{U_1} - \tilde{F}_{W_t} X_0, \\ e_\infty &\mapsto e_\infty, & U_1 &\mapsto W_1 = W_t - \tilde{F}_{W_t}, & \tilde{F}_{U_1} &\mapsto -\tilde{F}_{W_t}. \end{aligned}$$

The meaning of these transformations is not clear to the author.

⁴A Hamiltonian vector field on $\mathcal{S}_V(\tilde{a})$ with respect to the symplectic form $\tilde{\omega}$ (28) and a Hamiltonian function $H(X_0, W_t, U_1)$ is the vector field

$$(\tilde{F}_{W_t} H_{U_1} - \tilde{F}_{U_1} H_{W_t}) \partial_{X_0} + (\tilde{F}_{U_1} H_{X_0} - \tilde{F}_{X_0} H_{U_1}) \partial_{W_t} + (\tilde{F}_{X_0} H_{W_t} - \tilde{F}_{W_t} H_{X_0}) \partial_{U_1},$$

where $(H_{X_0}, H_{W_t}, H_{U_1}) = DH$ is the vector of partial derivatives of H .

5 Appendix: Painlevé equations as isomonodromic deformations of 3×3 systems

This section exposes how to derive the Fricke formula for the character variety of P_{VI} and for the modular group action on it as the space of Stokes data corresponding to isomonodromic deformations of 3×3 -systems in Okubo and Birkhoff canonical forms. Some of this can be also found in a bit different form in the article of Boalch [Boa05] (sections 2 and 3). Furthermore, we describe the confluence in these systems and show how the Stokes data of the limit system $\epsilon = 0$ are connected with those for $\epsilon \neq 0$ (Figure 10), providing thus another derivation of the character variety of P_V (23) and of the formulas of the birational change of variables Φ (26).

5.1 Systems in Okubo and Birkhoff forms

The sixth Painlevé equation P_{VI} governs also isomonodromic deformations 3×3 linear systems in *Okubo form*

$$\left(xI - \begin{pmatrix} 0 & \\ & t \end{pmatrix}\right) \frac{d\psi}{dx} = [B(t) + \lambda I]\psi, \quad (32)$$

where the matrix $B(t)$ can be written in the following form

$$B(t) = \begin{pmatrix} \vartheta_0 & w_0 u_t z_t - z_t & w_0 u_1 z_1 - z_1 \\ w_t u_0 z_0 - z_0 & \vartheta_t & w_t u_1 z_1 - z_1 \\ w_1 u_0 z_0 - z_0 & w_1 u_t z_t - z_t & \vartheta_1 \end{pmatrix}, \quad w_i = \frac{z_i + \vartheta_i}{u_i z_i}, \quad z_i \neq 0,$$

where ϑ_i are the parameters, and the matrix $B(t)$ has eigenvalues

$$0, \quad -\kappa_1 = \frac{1}{2}(\vartheta_0 + \vartheta_t + \vartheta_1 - \vartheta_\infty), \quad -\kappa_2 = \frac{1}{2}(\vartheta_0 + \vartheta_t + \vartheta_1 + \vartheta_\infty).$$

The system (32) can be obtained from the Garnier system (9) by the addition⁵ of $\frac{1}{2}(\frac{\vartheta_0}{x} + \frac{\vartheta_t}{x-t} + \frac{\vartheta_1}{x-1})I$ to $A(x, t)$, followed by the Katz's operation of middle convolution mc_λ with a generic parameter λ different from 0, κ_1, κ_2 [HF07] (see also [Boa05, Maz02]). One may also replace $B(t)$ by the conjugated matrix

$$\begin{pmatrix} z_0 & & \\ & z_t & \\ & & z_1 \end{pmatrix} B(t) \begin{pmatrix} z_0 & & \\ & z_t & \\ & & z_1 \end{pmatrix}^{-1} = \begin{pmatrix} \vartheta_0 & \frac{z_0 + \vartheta_0}{u_0} u_t - z_0 & \frac{z_0 + \vartheta_0}{u_0} u_1 - z_0 \\ \frac{z_t + \vartheta_t}{u_t} u_0 - z_t & \vartheta_t & \frac{z_t + \vartheta_t}{u_t} u_0 - z_t \\ \frac{z_1 + \vartheta_1}{u_1} u_0 - z_1 & \frac{z_1 + \vartheta_1}{u_1} u_t - z_1 & \vartheta_1 \end{pmatrix}.$$

Another isomonodromic problem that will be considered here is that of the system in a *Birkhoff canonical form*

$$\xi^2 \frac{dy}{d\xi} = \left[\begin{pmatrix} 0 & \\ & t \end{pmatrix} + \xi B(t) \right] y, \quad (33)$$

which is dual to the Okubo system (32) through the Laplace transform

$$y(\xi) = \xi^{-1-\lambda} \int_0^\infty \psi(x) e^{-\frac{x}{\xi}} dx, \quad |\arg(\xi) - \arg(x)| < \frac{\pi}{2}. \quad (34)$$

All three kinds of systems (9), (32), (33), and their isomonodromy problems are essentially equivalent (at least on a Zariski open set of irreducible systems (9)). Under an additional assumption that no ϑ_i is integral, the condition on (generalized) isomonodromicity of the each of the above linear systems is equivalent to the Painlevé equation $P_{VI}(\vartheta)$ [HF07].

⁵Corresponding to the gauge transformation $\phi \mapsto x^{-\frac{\vartheta_0}{2}} (x-t)^{-\frac{\vartheta_t}{2}} (x-1)^{-\frac{\vartheta_1}{2}} \phi$.

Notation 22. The elements of all 3×3 matrices will be indexed by $(0, t, 1)$ rather than $(1, 2, 3)$, corresponding to the eigenvalues of the matrix $\begin{pmatrix} 0 & t \\ & 1 \end{pmatrix}$. As before, the triple of indices (i, j, k) will always denote a permutation of $(0, t, 1)$, and (i, j, k, l) will denote a permutation of $(0, t, 1, \infty)$.

5.2 Stokes matrices of the Birkhoff system

The Birkhoff system (33) posses a canonical formal solution

$$\hat{Y}(\xi, t) = \hat{T}(\xi, t) \begin{pmatrix} \xi^{\vartheta_0} & & \\ & e^{-\frac{t}{\xi} \xi^{\vartheta_t}} & \\ & & e^{-\frac{1}{\xi} \xi^{\vartheta_1}} \end{pmatrix},$$

with $\hat{T}(\xi, t)$ an invertible formal series in ξ (with coefficients locally analytic in $t \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}$), which is unique up to right multiplication by an invertible diagonal matrix, and unique if one demands that $\hat{T}(0, t) = I$ [Sch01]. It is well known that this series is Borel summable in each non-singular direction α (we remind that a direction $\alpha \in \mathbb{R}$ is singular for the system (33) if $(i - j) \in e^{i\alpha} \mathbb{R}^+$ for some $i, j \in \{0, t, 1\}$, $i \neq j$). This means that for each non-singular direction α , there is an associated canonical fundamental matrix solution

$$Y_\alpha(\xi, t) = T_\alpha(\xi, t) \begin{pmatrix} \xi^{\vartheta_0} & & \\ & e^{-\frac{t}{\xi} \xi^{\vartheta_t}} & \\ & & e^{-\frac{1}{\xi} \xi^{\vartheta_1}} \end{pmatrix}, \quad |\arg(\xi) - \alpha| < \pi, \quad (35)$$

where T_α is the Borel sum in ξ of \hat{T} in the direction α .⁶ This solution does not depend on α as long as α does not cross any singular direction [Bal00, IY08, MR88, MR91].

Let us restrict to $\alpha \in]-\pi, \pi[$, and suppose for a moment that $0, t, 1$ are not collinear, i.e. there is six distinct singular rays $(i - j)\mathbb{R}^+$. When α crosses such a singular direction (in clockwise sense) the corresponding sectoral basis Y_α changes in a way that corresponds to multiplication by a constant (with respect to ξ) invertible matrix, so called *Stokes matrix*,

$$S_{ij} = I + s_{ij} E_{ij}, \quad (36)$$

where E_{ij} denotes the matrix with 1 at the position (i, j) and zero elsewhere. For the singular ray $-\mathbb{R}^+$, one needs to take in account also the jump in the argument of ξ between $-\pi$ and π , therefore the change of basis is provided by a matrix $\bar{N} S_{01}$, where \bar{N} is the *formal monodromy* of \hat{Y} :

$$\bar{N} = \begin{pmatrix} e_0^2 & & \\ & e_t^2 & \\ & & e_1^2 \end{pmatrix}, \quad \text{where } e_j := e^{\pi i \vartheta_j}. \quad (37)$$

See Figure 7 (a).

By definition, the (generalized) *isomonodromicity* of a family (33) demands that the Stokes matrices are independent of t .

Since in general the formal transformation \hat{T} , and therefore also the collection of the sectoral bases Y_α , are unique only up to right multiplication by invertible diagonal matrices, the collection of the Stokes matrices S_{ij} is defined only up a simultaneous conjugation by diagonal matrices. The obvious invariants with respect to such conjugation are

$$s_{0t} s_{t0}, \quad s_{t1} s_{1t}, \quad s_{10} s_{01}, \quad s_{0t} s_{t1} s_{10}, \quad s_{1t} s_{t0} s_{01}, \quad (38)$$

⁶The Borel sum T_α is given by the Laplace integral

$$T_\alpha(\xi, t) = \frac{1}{\xi} \int_0^{+\infty e^{i\alpha}} U(x, t) e^{-\frac{x}{\xi}} dx,$$

where $U(x, t) = \sum_{k=0}^{+\infty} \frac{T_k(t)}{k!} x^k$ is the formal Borel transform of $\xi \hat{T}(\xi, t) = \sum_{k=0}^{+\infty} T_k(t) \xi^{k+1}$.

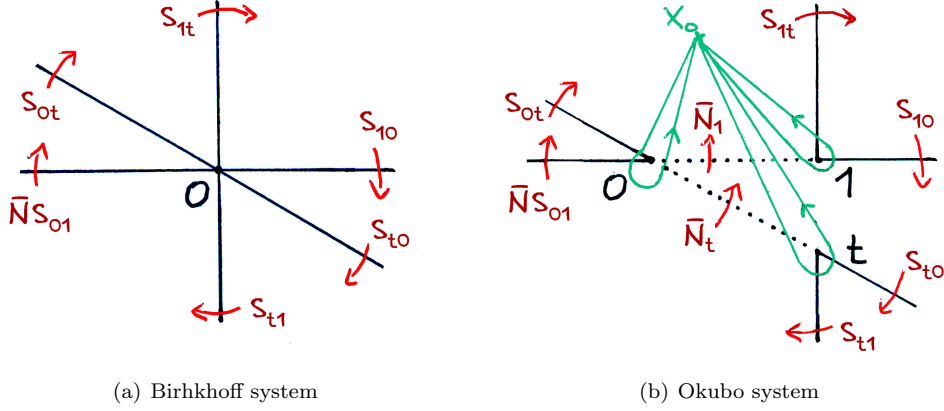


Figure 7: Singular directions and Stokes matrices.

subject to the relation

$$s_{0t}s_{t0} \cdot s_{t1}s_{1t} \cdot s_{10}s_{01} - s_{0t}s_{t1}s_{10} \cdot s_{1t}s_{t0}s_{01} = 0. \quad (39)$$

5.3 Monodromy of the Okubo system

Now consider the Okubo system (32), where we chose for simplicity

$$\lambda = 0.^7$$

Corresponding to the canonical sectoral solutions bases $Y_\alpha = (Y_{\alpha,ij})_{i,j}$ of (33), there are canonical sectoral solutions bases⁸ $\Psi_\alpha = (\Psi_{\alpha,ij})_{i,j}$ of (32), related to Y_α by the Laplace transform (34)

$$Y_{\alpha,ij}(\xi, t) = \frac{1}{\xi} \int_j^{+\infty e^{i\alpha}} \Psi_{\alpha,ij}(x, t) e^{-\frac{x}{\xi}} dx, \quad i, j \in \{0, t, 1\}.$$

The sectors on which they are defined (see Figure 7 (b)) are the different components of the complement in \mathbb{C} of

$$\bigcup_{i,j \in \{0,t,1\}} (i + (i-j)\mathbb{R}^+) \cup [0, 1] \cup [0, t].$$

When crossing one of the rays $i + (i-j)\mathbb{R}^+$ in clockwise sense the basis Ψ_α changes by the same Stokes matrix S_{ij} as before, except for the ray $-\mathbb{R}^+$, where it changes by $\bar{N}S_{01}$ (cf. [Kli15]). When crossing the segments $[0, t]$ the basis changes by \bar{N}_t , and on $[0, 1]$ by \bar{N}_1 , where

$$\bar{N}_i = I + (e_i^2 - 1)E_{ii}, \quad e_i = e^{\pi i \vartheta_i},$$

is the monodromy matrix of $\begin{pmatrix} x^{\vartheta_0} & \\ & (x-t)^{\vartheta_t} \\ & & (x-1)^{\vartheta_1} \end{pmatrix}$ around the point $i \in \{0, t, 1\}$, and $\bar{N} = \bar{N}_1 \bar{N}_t \bar{N}_0$ (37). See Figure 7 (b).

⁷For this choice of $\lambda = 0$ the system (32) is reducible but it does not matter here.

⁸ The fundamental matrix solution Ψ_α is also given by the convolution integral

$$\Psi_{\alpha,ij}(x, t) = \frac{1}{\Gamma(\vartheta_j + \lambda)} \int_j^x U_{ij}(z-j, t)(x-z)^{\vartheta_j + \lambda - 1} dz,$$

where $U(x, t) = \sum_{k=0}^{+\infty} \frac{T_k(t)}{k!} x^k$ is the formal Borel transform of $\xi \hat{T}(\xi, t) = \sum_{k=0}^{+\infty} T_k(t) \xi^k$ [Sch85, Kli15].

Fixing a base-point x_0 and three simple loops in positive direction around the points $0, t, 1$ respectively, such that their composition gives a simple loop around the infinity in negative direction as in see Figure 7 (b), let \bar{M}_i be the respective monodromy matrices:

$$\bar{M}_0 = \bar{N}_0 S_{01} S_{0t}, \quad \bar{M}_t = \bar{N}_1^{-1} S_{t0} S_{t1} \bar{N}_t \bar{N}_1, \quad \bar{M}_1 = S_{1t} S_{10} \bar{N}_1,$$

determined up to simultaneous conjugation in $\text{GL}_3(\mathbb{C})$. We have

$$\text{tr}(\bar{M}_i) = e_i^2 + 2, \quad i \in \{0, t, 1\}.$$

Denoting

$$X_i = \frac{\text{tr}(\bar{M}_j \bar{M}_k) - 1}{e_j e_k},$$

we have

$$X_0 = \frac{e_t^2 + e_1^2 + e_1^2 s_{t1} s_{1t}}{e_t e_1}, \quad X_t = \frac{e_0^2 + e_1^2 + e_0^2 s_{10} s_{01}}{e_0 e_1}, \quad X_1 = \frac{e_0^2 + e_t^2 + e_0^2 s_{0t} s_{t0}}{e_0 e_t}. \quad (40)$$

The monodromy around all the three points equals

$$\begin{aligned} \bar{M}_\infty^{-1} &= \bar{M}_1 \bar{M}_t \bar{M}_0 = S_{1t} S_{10} S_{t0} S_{t1} \bar{N} S_{01} S_{0t} \\ &= \begin{pmatrix} e_0^2 & e_0^2 s_{0t} & e_0^2 s_{01} \\ e_0^2 s_{t0} & e_t^2 + e_0^2 s_{t0} s_{0t} & e_1^2 s_{t1} + e_0^2 s_{t0} s_{01} \\ e_0^2 s_{10} + e_0^2 s_{1t} s_{t0} & e_t^2 s_{1t} + e_0^2 s_{10} s_{0t} + e_0^2 s_{1t} s_{t0} s_{0t} & e_1^2 + e_1^2 s_{1t} s_{t1} + e_0^2 s_{10} s_{01} + e_0^2 s_{1t} s_{t0} s_{01} \end{pmatrix} \end{aligned}$$

We know that its eigenvalues are 1, $e^{-2\pi i \kappa_1} = \frac{e_0 e_t e_1}{e_\infty}$ and $e^{-2\pi i \kappa_2} = e_0 e_t e_1 e_\infty$. Expressing the coefficients of the linear term E and the quadratic term E' of the characteristic polynomial of \bar{M}_∞^{-1} leads to

$$e_t e_1 X_0 + e_0 e_1 X_t + e_0 e_t X_1 + e_0^2 s_{1t} s_{t0} s_{01} - e_0^2 - e_t^2 - e_1^2 = 1 + \frac{e_0 e_t e_1}{e_\infty} + e_0 e_t e_1 e_\infty := E, \quad (41)$$

$$\begin{aligned} e_0^2 e_t e_1 X_0 + e_t^2 e_0 e_1 X_t + e_1^2 e_0 e_t X_1 - e_0^2 e_1^2 s_{0t} s_{t1} s_{10} - e_0^2 e_t^2 - e_t^2 e_1^2 - e_1^2 e_0^2 &= \\ = \frac{e_0 e_t e_1}{e_\infty} + e_0 e_t e_1 e_\infty + e_0^2 e_t^2 e_1^2 := E'. \end{aligned} \quad (42)$$

Inserting the expression for $s_{1t} s_{t0} s_{01}$ (41) and for $s_{0t} s_{t1} s_{10}$ (42) into the relation (39) gives the *Fricke relation* (14)

$$X_0 X_t X_1 + X_0^2 + X_t^2 + X_1^2 - \theta_0 X_0 - \theta_t X_t - \theta_1 X_1 + \theta_\infty,$$

with

$$\begin{aligned} \theta_i &= \frac{e_j^2 + e_k^2 + E}{e_j e_k} + \frac{E'}{e_i^2 e_j e_k} = a_i a_\infty + a_j a_k, \\ \theta_\infty &= 1 + \frac{E}{e_0} + \frac{E}{e_t} + \frac{E}{e_1} + \frac{E'}{e_0 e_t} + \frac{E'}{e_t e_1} + \frac{E'}{e_1 e_0} = a_0 a_t a_1 a_\infty + a_0^2 + a_t^2 + a_1^2 + a_\infty^2 - 4. \end{aligned}$$

The line

$$\{X_k = \frac{e_i}{e_j} + \frac{e_j}{e_i}, \quad e_i X_i + e_j X_j = a_\infty + e_i e_j a_k\}$$

of Proposition 9 corresponds to $s_{ij} = 0$ in (42), while

$$\{X_k = \frac{e_i}{e_j} + \frac{e_j}{e_i}, \quad e_i X_j + e_j X_i = a_k + e_i e_j a_\infty\}$$

corresponds to $s_{ji} = 0$ in (41), where (i, j, k) is a cyclic permutation of $(0, t, 1)$.

We can also derive the induced action of the braids β_{0t} and β_{t1} (Figure 3.4) on the Stokes matrices S_{ij} .

Proof of Proposition 10. The action of the braids β_{0t} and β_{t1} on the Stokes matrices is obtained by:

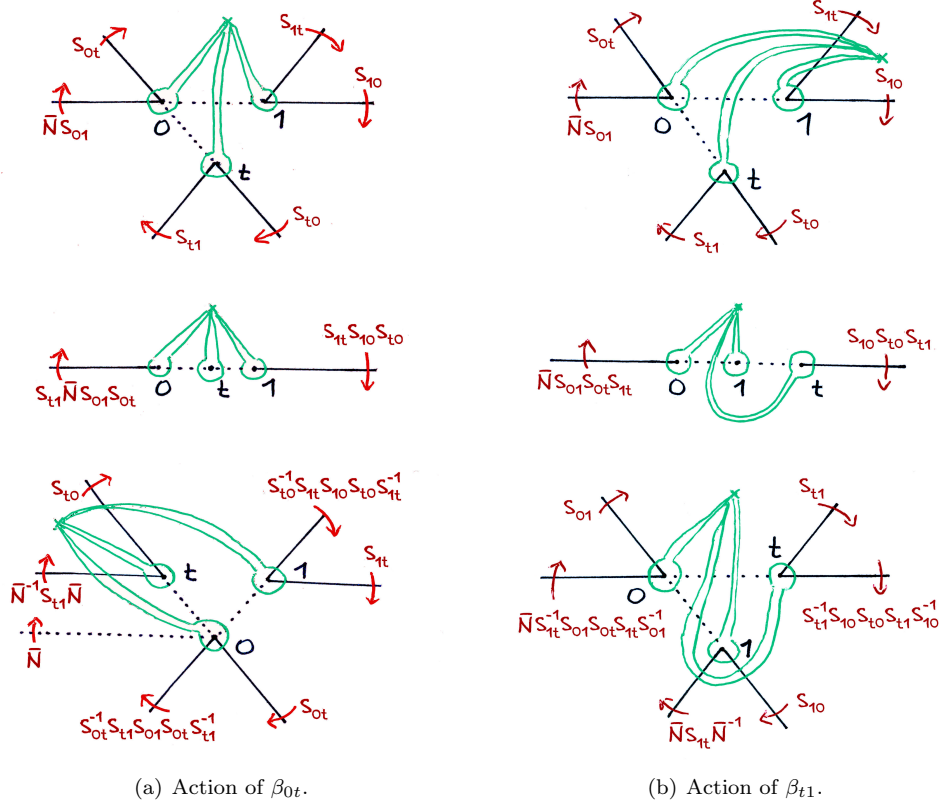


Figure 8: Braid actions on the Stokes matrices.

- 1) Tracing the connection matrices of the Okubo system (32) as the two corresponding points turn around each other according to the braid β_{0t} , resp. β_{t1} , and see how they change when the three points $0, t, 1$ align. See Figure 8. We use the fact that $S_{ij}S_{kl} = S_{kl}S_{ij}$ if $j \neq k$ and $l \neq i$.
- 2) Swapping the names of the points $0 \leftrightarrow t$, resp. $t \leftrightarrow 1$. This permutes also the corresponding positions of all the matrices by $P_{0t} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, resp. $P_{t1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

We found that the action on the Stokes matrices is given (up to simultaneous conjugation by diagonal matrices) by

$$\begin{array}{ll}
 \beta_{0t} : & \begin{array}{l} \bar{N} \mapsto P_{0t} \bar{N} P_{0t}, \\ S_{t0} \mapsto P_{0t} S_{0t} P_{0t}, \\ S_{t0}^{-1} S_{1t} S_{10} S_{t0} S_{1t}^{-1} \mapsto P_{0t} S_{1t} P_{0t}, \\ S_{1t} \mapsto P_{0t} S_{10} P_{0t}, \\ S_{0t} \mapsto P_{0t} S_{t0} P_{0t}, \\ S_{0t}^{-1} S_{t1} S_{01} S_{0t} S_{t1}^{-1} \mapsto P_{0t} S_{t1} P_{0t}, \\ \bar{N}^{-1} S_{t1} \bar{N} \mapsto P_{0t} S_{01} P_{0t}, \end{array} \\
 \beta_{t1} : & \begin{array}{l} \bar{N} \mapsto P_{t1} \bar{N} P_{t1}, \\ S_{t1} \mapsto P_{t1} S_{1t} P_{t1}, \\ S_{t1}^{-1} S_{10} S_{t0} S_{t1} S_{10}^{-1} \mapsto P_{t1} S_{10} P_{t1}, \\ S_{10} \mapsto P_{t1} S_{t0} P_{t1}, \\ \bar{N} S_{1t} \bar{N}^{-1} \mapsto P_{t1} S_{t1} P_{t1}, \\ S_{1t}^{-1} S_{01} S_{0t} S_{1t} S_{01}^{-1} \mapsto P_{t1} S_{01} P_{t1}, \\ S_{01} \mapsto P_{t1} S_{0t} P_{t1}, \end{array}
 \end{array}$$

from which the corresponding action of g_{0t} , resp. g_{t1} , on the invariant elements (38) can be easily expressed, and subsequently re-expressed in terms of the coordinates X_0, X_t, X_1 (40). \square

5.4 Confluence of the Birkhoff systems and their character varieties

The substitution (18) in the Birkhoff system (33) and a conjugation by $Q = \begin{pmatrix} \epsilon t & 1 & -1 \\ & \epsilon t & \end{pmatrix}$, corresponding to the change of variable $\tilde{y} = Qy$, gives a parametric family of isomonodromic systems

$$\xi^2 \frac{d\tilde{y}}{d\xi} = \left[\begin{pmatrix} 0 & 1+\epsilon t & 1 \\ & & \end{pmatrix} + \xi \tilde{B}(t, \epsilon) \right] \tilde{y}, \quad (43)$$

with

$$\begin{aligned} \tilde{B} = QBQ^{-1} &= \begin{pmatrix} \vartheta_0 & \epsilon t(w_0 u_t z_t - z_t) & -w_0 u_0 z_0 - z_1 - z_t \\ \frac{1}{\epsilon t}(w_t - w_1)u_0 z_0 & \vartheta_t + z_t - w_1 u_t z_t & -\frac{1}{\epsilon t}(w_t - w_1)u_0 z_0 \\ w_1 u_0 z_0 - z_0 & \epsilon t(w_1 u_t z_t - z_t) & \vartheta_1 - z_t + w_1 u_t z_t \end{pmatrix} \\ &= \begin{pmatrix} \vartheta_0 & u_1 \tilde{z}_1 \frac{(z_0 + \vartheta_0)}{u_0 z_0} - \tilde{z}_1 - \epsilon t(\kappa_2 + \vartheta_0) & -\kappa_2 - \vartheta_0 \\ -\tilde{b}_{t1} & 1 + \tilde{\vartheta}_1 + \kappa_2 + b_{10} & \tilde{b}_{t1} \\ \tilde{b}_{10} & t - \epsilon t[1 + \tilde{\vartheta}_1 + \kappa_2 - z_0 + \frac{\tilde{z}_1 + t}{u_1 \tilde{z}_1}] & -\kappa_2 - \tilde{b}_{10} \end{pmatrix}, \end{aligned}$$

where $\kappa_2 = -\frac{\vartheta_0 + \tilde{\vartheta}_1 + 1 + \vartheta_\infty}{2}$, and

$$\tilde{b}_{10} = \frac{u_0 z_0}{u_1 \tilde{z}_1} (\tilde{z}_1 + t - \epsilon t(1 + \tilde{\vartheta}_1)) - z_0, \quad \tilde{b}_{t1} = -\frac{u_0 z_0}{u_1 \tilde{z}_1} \frac{u_0 z_0 (\tilde{z}_1 + t - \epsilon t(1 + \tilde{\vartheta}_1)) - u_1 \tilde{z}_1 (z_0 - 1 - \tilde{\vartheta}_1 - \kappa_2)}{u_1 \tilde{z}_1 - \epsilon t u_0 z_0}.$$

When $\epsilon \neq 0$ the irregular singular point at the origin is non-resonant and the local description of the Stokes phenomenon is just the same as in the precedent section with six Stokes matrices S_{ij} . But for $\epsilon = 0$ the singularity becomes resonant and the description changes.

For $|\epsilon t|$ small, there is a formal transformation $\tilde{y} = \hat{T}(x, \epsilon) \begin{pmatrix} \tilde{y}' \\ \tilde{y}'' \end{pmatrix}$, $\tilde{y}' \in \mathbb{C}$, $\tilde{y}'' \in \mathbb{C}^2$, written as a formal power series in x with coefficients analytic in ϵ , that splits the system in two diagonal blocks, one corresponding to the eigenvalue 0, other corresponding to the other eigenvalues $\{1 + \epsilon t, 1\}$ (cf. [Bal00]):

$$\xi^2 \frac{d\tilde{y}'}{d\xi} = \xi \vartheta_0 \tilde{y}', \quad (44)$$

$$\xi^2 \frac{d\tilde{y}''}{d\xi} = \left[\begin{pmatrix} 1+\epsilon t & 1 \\ 0 & 1 \end{pmatrix} + \xi \tilde{B}''(\epsilon) + O(\xi^2) \right] \tilde{y}'', \quad (45)$$

where $\tilde{B}'' = \begin{pmatrix} \tilde{b}_{tt} & \tilde{b}_{t1} \\ \tilde{b}_{1t} & \tilde{b}_{11} \end{pmatrix}$ is the submatrix of \tilde{B} . This formal transformation \hat{T} is Borel summable in all directions except of $\pm \mathbb{R}^+$ and $\pm(1 + \epsilon t)\mathbb{R}^+$. Therefore it possesses Borel sums on the four sectors, overlapping on the singular directions, out of which only those on the two large sectors persist to the limit $\epsilon \rightarrow 0$.

Confluence of eigenvalues in the subsystem (45). The phenomenon of confluence of eigenvalues in 2×2 parametric systems at an irregular singular point of Poincaré rank 1 was studied previously by the author [Kli14]. This paragraph applies some of the results to the system (45).

The matrix of the right side of the system has its eigenvalues equal to

$$\lambda^{(0)} + \xi \lambda^{(1)} \pm \sqrt{\alpha^{(0)} + \xi \alpha^{(1)}} \pmod{\xi^2},$$

where

$$\begin{aligned} \lambda^{(0)} &= 1 + \frac{\epsilon t}{2}, & \lambda^{(1)} &= \frac{\tilde{b}_{tt} + \tilde{b}_{11}}{2} = \frac{1 + \tilde{\vartheta}}{2}, \\ \alpha^{(0)} &= \left(\frac{\epsilon t}{2}\right)^2, & \alpha^{(1)} &= \tilde{b}_{1t} + \frac{\epsilon t(\tilde{b}_{tt} - \tilde{b}_{11})}{2} = \epsilon t \frac{\vartheta_t - \vartheta_1}{2} = t - \epsilon t \frac{1 + \tilde{\vartheta}_1}{2}, \end{aligned}$$

constitute the formal invariants of the system. In [Kli14], it has been shown that (45) possess a fundamental matrix solutions of the form

$$\tilde{Y}_{\bullet}'' = R_{\bullet}''(\xi, t, \epsilon) \cdot e^{-\frac{\lambda^{(0)}}{\xi}} \xi^{\lambda^{(1)}} \begin{pmatrix} (\alpha^{(0)} + \xi \alpha^{(1)})^{-\frac{1}{4}} & \\ & (\alpha^{(0)} + \xi \alpha^{(1)})^{\frac{1}{4}} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{\Theta} & \\ & e^{-\Theta} \end{pmatrix},$$

where

$$\Theta(\xi, \epsilon, t) = \int_{\infty}^{\xi} \frac{\sqrt{\alpha^{(0)} + \xi \alpha^{(1)}}}{\zeta^2} d\zeta = \begin{cases} -\frac{\sqrt{\alpha^{(0)} + \xi \alpha^{(1)}}}{\xi} - \frac{\alpha^{(1)}}{2\sqrt{\alpha^{(0)}}} \log \frac{\sqrt{\alpha^{(0)} + \xi \alpha^{(1)}} + \sqrt{\alpha^{(0)}}}{\sqrt{\alpha^{(0)} + \xi \alpha^{(1)}} - \sqrt{\alpha^{(0)}}}, & \epsilon \neq 0, \\ -\frac{2\alpha^{(0)}}{\xi}, & \epsilon = 0, \end{cases}$$

and R_{\bullet}'' , $\bullet = I_{\pm}, O$, are invertible analytic transformations defined on certain domains in the ξ -space. These domains are delimited by the so called Stokes curves (in the sense of exact WKB analysis [KT06]): the separatrices of the real phase portrait of the vector field⁹

$$e^{i\omega} \frac{\xi^2}{2\sqrt{\alpha^{(0)} + \xi \alpha^{(1)}}} \frac{\partial}{\partial \xi}, \quad \text{with some } \omega \in]-\frac{\pi}{2}, \frac{\pi}{2}[$$

emanating either from ∞ or from the “turning point” at $\xi = -\frac{\alpha^{(0)}}{\alpha^{(1)}}$, if $\epsilon \neq 0$.

There are two kinds of such sectoral domains, whose shape in the coordinate $\frac{\xi}{\alpha^{(1)}}$ depends only on a parameter $\mu = \frac{\alpha^{(0)}}{(\alpha^{(1)})^2} = \left(\frac{\epsilon}{2-\epsilon(1+\vartheta_1)}\right)^2$:

- A pair of inner domains I_{\pm} for $\epsilon \neq 0$: these are sectors at 0 of radius proportionate to $\mu \sim \frac{\epsilon^2}{4}$, separated one from another by the singular directions $\pm \epsilon t \mathbb{R}^+$. They disappear at the limit. The connection matrices between \tilde{Y}_{I_+}'' and \tilde{Y}_{I_-}'' are given by $S_{1t}'' = \begin{pmatrix} 1 & 0 \\ s_{1t} & 1 \end{pmatrix}$, $S_{t1}'' = \begin{pmatrix} 1 & s_{t1} \\ 0 & 1 \end{pmatrix}$, submatrices of the Stokes matrices S_{1t} , S_{t1} (36).
- An outer domain O covering a complement of $I_+ \cup I_-$ in a disc of a fixed radius with a cut in the direction $\frac{\alpha^{(0)}}{\alpha^{(1)}} \mathbb{R}^+ \sim \epsilon^2 t \mathbb{R}^+$. We are mainly interested in the limit when $\epsilon \rightarrow 0$ along the sequences $\frac{1}{\epsilon} \in \frac{1}{\epsilon_0} \pm 2\mathbb{N}$, hence the cut will be always in the direction $t \mathbb{R}^+$. The connection matrix on this cut is $\tilde{S}_t'' = \begin{pmatrix} X_0 & -i \\ -i & 0 \end{pmatrix}$. See Figure 9.

Returning now to the full system (43), one must intersect the domains I_{\pm}, O with the sectors of the Borel summability of the transformation \hat{T} . The full picture is therefore that of Figure 10. There are six inner Stokes matrices S_{ij} (36) between the canonical solutions on the inner domains, and five outer Stokes matrices between the canonical solutions on the outer domains. Only three of the outer domains persist to the limit $\epsilon \rightarrow 0$ together with the associated Stokes matrices:

$$\tilde{S}_0 = \begin{pmatrix} 1 & \tilde{s}_{0t} & \tilde{s}_{01} \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \tilde{S}_t = \begin{pmatrix} 1 & & \\ X_0 & -i & \\ -i & 0 & \end{pmatrix}, \quad \tilde{S}_1 = \begin{pmatrix} 1 & & \\ \tilde{s}_{t0} & 1 & \\ \tilde{s}_{10} & & 1 \end{pmatrix}, \quad \tilde{N} = \begin{pmatrix} e_0^2 & & \\ & e_t e_1 & \\ & & e_t e_1 \end{pmatrix}. \quad (46)$$

The connection matrices between the canonical bases on the inner and outer domains are provided by:

$$C_+ = \begin{pmatrix} 1 & & \\ & 1 & \frac{1}{s_{1t}} \\ & 0 & -i \frac{e_t}{e_1 s_{1t}} \end{pmatrix}, \quad C_- = \begin{pmatrix} 1 & & \\ 0 & \frac{e_t}{e_1 s_{1t}} & \\ i & -i \frac{e_t^2}{e_1^2 s_{1t}} & \end{pmatrix}. \quad (47)$$

⁹This vector field is defined only on the Riemann surface of $\sqrt{\alpha^{(0)} + \xi \alpha^{(1)}}$ but the foliation by its real trajectories projects well onto the ξ -space.

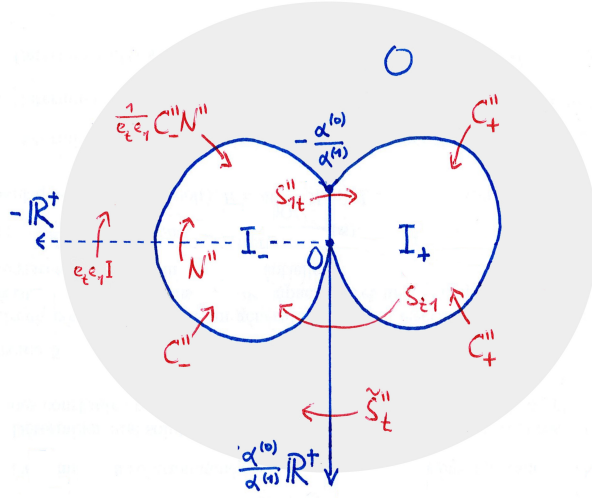


Figure 9: The inner and outer domains, and the connection matrices of the system (45).

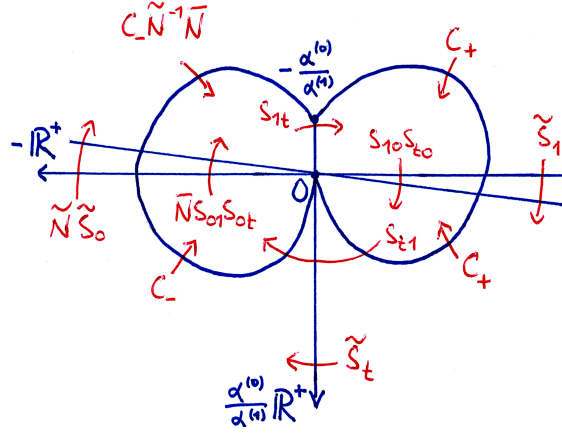


Figure 10: The Stokes matrices of the confluent system, N (37), S_{ij} (15), \tilde{S}_i (46), C_{\pm} (47).

Lemma 23. *The coefficients of the outer Stokes matrices are equal to*

$$\begin{aligned} \tilde{s}_{0t} &= s_{0t} + s_{01}s_{1t}, & \tilde{s}_{01} &= -i\frac{e_1}{e_t}s_{0t}, \\ \tilde{s}_{t0} &= s_{t0} + \frac{s_{10}}{s_{1t}}, & \tilde{s}_{10} &= -i\frac{e_t}{e_1}\frac{s_{10}}{s_{1t}}. \end{aligned} \quad (48)$$

Proof. We have $\tilde{S}_0 = C_- \tilde{N}^{-1} \tilde{N} S_{01} S_{0t} \tilde{N}^{-1} \tilde{N} (C_-)^{-1}$, $\tilde{S}_1 = C_+ S_{10} S_{t0} (C_+)^{-1}$, see Figure 10. \square

Remark 24. The inner Stokes matrices S_{ij} are considered only up to conjugation by diagonal matrices $\begin{pmatrix} d_0 & & \\ & d_t & \\ & & d_1 \end{pmatrix}$. This corresponds through (48) to conjugation of the outer Stokes matrices \tilde{S}_i by $\begin{pmatrix} d_0 & & \\ & d_t & \\ & & d_t \end{pmatrix}$.

The new variables (X_0, W_t, U_1) (22) are defined by

$$X_0 = X_0, \quad W_t = ie_0 \tilde{s}_{10} \tilde{s}_{0t} + \frac{e_t e_1}{e_0}, \quad U_1 = e_0 \tilde{s}_{10} \tilde{s}_{01} + e_0. \quad (49)$$

They are invariant with respect to the conjugation of Remark 24.

The monodromy matrix of an outer solution around the origin is given by

$$\tilde{M}_\infty^{-1} = \tilde{S}_1 \tilde{S}_t \tilde{N} \tilde{S}_0 = \begin{pmatrix} e_0^2 & e_0^2 \tilde{s}_{0t} & e_0^2 \tilde{s}_{01} \\ e_0^2 \tilde{s}_{t0} & e_t e_1 X_0 + e_0^2 \tilde{s}_{t0} \tilde{s}_{0t} & -i e_t e_1 + e_0^2 \tilde{s}_{t0} \tilde{s}_{01} \\ e_0^2 \tilde{s}_{10} & -i e_t e_1 + e_0^2 \tilde{s}_{10} \tilde{s}_{0t} & e_0^2 \tilde{s}_{10} \tilde{s}_{01} \end{pmatrix},$$

and we know that its eigenvalues are again 1, $e^{-2\pi i \kappa_1} = \frac{e_0 e_t e_1}{e_\infty}$ and $e^{-2\pi i \kappa_2} = e_0 e_t e_1 e_\infty$. Expressing the coefficients of the linear and the quadratic term of the characteristic polynomial of \tilde{M}_∞^{-1} gives therefore

$$e_t e_1 X_0 + e_0 U_1 + e_0^2 \tilde{s}_{0t} \tilde{s}_{t0} = 1 + \frac{e_0 e_t e_1}{e_\infty} + e_0 e_t e_1 e_\infty := E,$$

$$e_0 e_t e_1 X_0 U_1 + e_0 e_t e_1 W_t + i e_0^2 e_t e_1 \tilde{s}_{t0} \tilde{s}_{01} = \frac{e_0 e_t e_1}{e_\infty} + e_0 e_t e_1 e_\infty + e_0^2 e_t^2 e_1^2 := E'.$$

Inserting these two identities into the obvious relation

$$\begin{aligned} 0 &= e_0^2 \cdot \tilde{s}_{01} \tilde{s}_{t0} \cdot \tilde{s}_{10} \tilde{s}_{0t} - e_0^2 \cdot \tilde{s}_{0t} \tilde{s}_{t0} \cdot \tilde{s}_{10} \tilde{s}_{01} \\ &= -i \tilde{s}_{01} \tilde{s}_{t0} (e_0 W_t + e_t e_1) - \tilde{s}_{0t} \tilde{s}_{t0} (e_0 U_1 - e_0^2) = 0 \end{aligned}$$

(49), gives the Fricke relation (23)

$$0 = X_0 W_t U_1 + W_t^2 + U_1^2 - \tilde{\theta}_0 X_0 - \tilde{\theta}_t W_t - \tilde{\theta}_1 U_1 + \tilde{\theta}_\infty,$$

with

$$\begin{aligned} \tilde{\theta}_0 &= e_t e_1, & \tilde{\theta}_t &= \frac{e_t e_1}{e_0} + \frac{E'}{e_0 e_t e_1} = a_\infty + e_t e_1 a_0, \\ \tilde{\theta}_\infty &= E + \frac{E'}{e_0^2} = 1 + e_t e_1 a_0 a_\infty + e_t^2 e_1^2, & \tilde{\theta}_1 &= e_0 + \frac{E}{e_0} = a_0 + e_t e_1 a_\infty. \end{aligned}$$

The formulas of Proposition 14 can be obtained from (48) and (40), (49). The singular line $L_0(a)$ (27) corresponds to $s_{1t} = 0$.

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